

A STUDY OF RECTANGULAR PLATES SUBJECTED  
TO NON-UNIFORM AXIAL COMPRESSION

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A STUDY OF RECTANGULAR PLATES SUBJECTED  
TO NON-UNIFORM AXIAL COMPRESSION

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## NOMENCLATURE

P	Applied Load
$\delta$	Out of Plane Deformation
$f_0$	Buckle Amplitude
$\lambda_0$	Buckle Width
A	Cross-Sectional Area of Plate
$P_c$	Buckling Load
u	$(F_0/\lambda)^2 = \delta^2$
$\sigma_c$	Buckling Stress
$\epsilon_c$	Buckling Strain
t	Thickness of the Sheet (Plate)
b	Width of the Sheet (Plate)
R	Radius of Curvature
$W(x,y), w(x,y)$	Deflection Function
$X(x), Y(y)$	Beam Functions
W	Total Load Acting on the Beam
E	Young's Modulus
I	Moment of Inertia
L	Length of Beam
$w_1$	Peak Value of the Distributed Load Acting on the Beam
n	Number of Half Waves in the x-direction
a	Length of the Plate
$\pi$	3.14159...



## NOMENCLATURE (Continued)

$\lambda$	$1/60 \frac{W}{EI}$
$N_x$	Normal Force Per Unit Length in the Middle Surface of the Plate in the x Direction
$N_o$	Peak Value of $N_x$
$\alpha_L$	Loading Profile Parameter
$\nu$	Poisson's Ratio
$K$	Buckling Coefficient
$a/b = \phi$	Plate Aspect Ratio
$D$	$\frac{Et^3}{12(1-\nu^2)}$ Plate Flexural Rigidity
$K_1, K_0$	Edge Restraint Coefficients Used by Walker
$q_0, \dots, q_6$	Galerkin Coefficients in Deflection Function Series
$\alpha, \beta$ $\beta_1, \beta_2$	Plate Lateral Edge Restraint Parameters
$A, B^*$	Arbitrary Deflection Amplitudes Used by Lundquist & Stowell
$\epsilon$	Restraint Coefficient of Lundquist & Stowell
$S_0$	Stiffness Per Unit Length of Elastic Restraining Medium
$Z$	Beam Deflection Developed by Nassar
$z_{\infty},$ $z_{\infty 0},$ $z_{0\infty},$ $z_{00}$	Deflection Function of a Beam Under a Given Lateral Load for Extreme Values of Lateral Restraint
$z_1, z_2$	Beam Deflection Functions
$z_{\text{trapezoidal}}$	Deflection Function for a Beam with a Trapezoidal Distribution of Lateral Load

## NOMENCLATURE (Continued)

$AR_{\infty\infty}$ $AR_{00}$ $A_{\beta\beta}$	Effective Aspect Ratio for a Plate with Equal Lateral Restraint
$C, C^*$	"Matching Point" Coefficient in the Rational Function Expression
$K_{\infty\infty}, K_{00}$ $K_{0\infty}, K_{\infty 0}$	Buckling Coefficient for Plates with Extreme Values of Lateral Restraint
$K_{\beta\beta}$	Buckling Coefficient for a Plate with Equal Lateral Restraints
$K_{\alpha\beta}$	Buckling Coefficient for a Plate with Unequal Lateral Restraints
$A_1, B_1, C_1$	Factors of Proportional
L.P.F.	Loading Profile Fraction
N	Numerical Correlation Factor Between Linearly Varying Axial Load Cases
$\alpha_L^*$	Redefined Loading Profile Parameter
$a_L$	Lessor Load Value in a Linear Distribution
$b_L$	Greater Load Value in a Linear Distribution

## SUMMARY

Since the mid-1800's, when Fairbairn and Hodgkinson discovered plate instability and conducted many tests on such structural elements, much work has been done on the subject. The activity in the area of theoretical analysis has been reviewed and summarized by previous workers; however, this is not so for the extensive literature on experimental studies. Thus, an in depth investigation into the techniques, developments and results of laboratory testing was made and the details are presented in the thesis. Several significant facts become clear. First, experimentalists like theoreticians have tended to ignore the practical issues of boundary restraint. They have devised no method of attaining any prescribed restraint condition other than simply supported or fixed and have made no effort to develop any technique of assessing the restraint in a realistic situation. As with boundary restraint so in general with load, the prime attention has been directed toward uniformity. However, over the years there has been progressive refinement in method and continual improvement in quality of test data generated and the agreement between the experimental observations made by different researchers is now qualitatively and quantitatively good. Moreover, in recent times it has been repeatedly demonstrated that under classical boundary conditions the agreement between analysis and experiment is excellent. This is, in large part, due to modern test techniques which, of course, depend upon the quality of current instrumentation. However, the fact that all results in which the agreement is established were

interpreted using the Southwell procedure to determine instability load level should not be overlooked.

The study made then gives very strong grounds for the contention that new investigations can depend solely upon either analysis or controlled experiment. Thus, since the former is the least expensive it will, a priori, in general take precedence. This is not to assert that there are not practical issues in realistic structures which will not and do not require experimental evaluation. This simple fact of economics thus sets the tone for the studies which are reported in the later portions of this thesis.

In the third chapter it is conclusively demonstrated that there are consistent and logical processes for choosing deflection functions, which enable the Raleigh Quotient to be applied with success to the issues of non-uniform distribution of compressive load and varying degree of edge rotational restraint. Very detailed calculations show positively that considerable latitude in function choice exists. Moreover, the agreement between all the solutions so generated and those derived on orthodox lines is excellent. This gave complete confidence that the methods could be extended to issues not previously resolved and so a broader range of load distributions and boundary restraint conditions was studied in a like manner. The numerous data generated were found to be expressible in powerful, concise and precise formulae. Such formulae are of unestimable value to designers and analysts, because they are infinitely more convenient than the usual profusion of charts and curves.

It is anticipated that when these results are taken in conjunction with those developed by other investigators for the frequencies of

vibrating plates they will lead to practical methods of assessing boundary restraint value for realistic structures. Such a step will clearly have great value to designers since this knowledge will lead to increased efficiency and reliability.



## CHAPTER I

### INTRODUCTION

I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Sir Isaac Newton

Such an area is that of the stability of columns and plates. Although the subject of elastic stability has been considered for some 230 years the bulk of our knowledge of the buckling of flat plates has been acquired within the past 50 years. In point of fact the majority of contributors to the field of study are still alive.

Attention has been paid mainly to the refinement of knowledge but unification and simplification have not received due attention. But these are facets of the utmost importance. Increasing simplification leads to wider use of the knowledge. This in turn leads to the development of more refined engineering devices in the more mundane applications.

Today we are faced with energy shortages, rapid depletion of world resources and increasing demands for the products of engineering creation. Efficiency in design and thus use of these scarce commodities is now a prime consideration. Thus in general structural engineering there is a need to move towards that prime requirement of aeronautical

structures, viz., the optimization of the strength to weight ratio. It is true that this move began with the advent of the iron-bridge, an advent which heralded great advances in bridge engineering. A beginning has been made in building construction but there is yet much to be done. There seems little doubt that economic and allied pressure will accelerate developments in this area. These developments will demand the use of knowledge not commonly associated with the pertinent disciplines. This does not pose a unique situation for in this age of specialization the engineer is often called upon to perform outside his sphere of specialized knowledge. It is in such instances that straightforwardness of concept and ready applicability of knowledge is of the greatest value. Thus over the years the various analytical and experimental techniques have been and are subject to constant scrutiny to achieve ever increasing simplicity without significant loss of accuracy. This is a most vital function of research engineers.

The current thesis is part of a concerted effort in this area. A detailed study of the experimental studies on plates was undertaken. This study demonstrated that the experimentalists and practising engineers experienced great difficulty in achieving prescribed loading and/or boundary restraint conditions. A prior study of the analytical contribution [1] shows clearly that the theoreticians have dealt mainly with uniform loading actions and simplified boundary restraint. Thus for one reason or another non-uniform loading actions particularly in combination with unequal boundary restraints has been largely ignored



by both theoreticians and experimentalists.

The definition or establishment of the degree of restraint in a practical situation is not easy. Neither is it, of course, for the case of columns. Recent researches [2,3] in the latter field have however been able to associate behavior in a destabilizing environment with that in a non-destabilizing environment and so establish a process for the determination of boundary restraint parameters. A fundamental step in this work was the establishment of an approximate but explicit relationship between the instability load levels and the boundary restraint parameters. The corresponding relationship between, for example, deflection under lateral load and the end fixity was, of course, more readily determinable. The success of this investigation [3] and the promising initial study of plate relationships published subsequently [4] lead to the thought that much progress might come from developing simplified relationships for plate buckling coefficients in terms of boundary restraint values. Such a development could conceivably lead to a practical method of assessment of edge restraint and most certainly bring the simplification necessary for common usage. This then was the motivation which led to the studies delineated in Chapters III and IV.

In Chapter III a simplified approach to the determination of the buckling coefficients for non-uniformly loaded plates with various boundary conditions is outlined. A number of cases of potential importance are treated. The solutions, whenever possible, are tested against prior work. It is demonstrated that the agreement is good.

In Chapter IV these various results are simplified and coordinated and generalized expressions of great simplicity and high accuracy developed. These expressions are readily usable both for direct application and in computerized optimization studies. At the same time they appear of such format that they might be associatable with the corresponding formulations for behavior under non-destabilizing forces when such expressions are developed.

## CHAPTER II

### A HISTORICAL SURVEY OF EXPERIMENTAL STUDIES ON THE STABILITY OF PLATES

For well over a century, experimental studies have been made on the stability of plates subject to axial compression. Although the specimens tested have been of various sizes, radii and materials; all of the tests have experienced a common difficulty, viz., that of achieving a desired set, or even an accurately definable set of boundary conditions. The theoretical solutions for the stability of plates are based upon boundary conditions which, in the main, can be approached but not realized experimentally. The ensuing predicament was aptly described by Redshaw [5] when he stated: "Broadly speaking, the tester takes the theorist to task for not allowing for practical variations while the theorist, perhaps more unreasonably, wishes the tester would use perfect plates and reproduce perfect boundary conditions."

The excellent review by Bulson [1] of the many theoretical studies made on plate stability has already been cited but scant attention has been paid to the numerous experimental studies. Thus to attain a balanced view of the shortcomings of our knowledge this study has been undertaken.

#### Early Investigations

Historically, the first experimental studies of the stability of

plates were those conducted by Fairbairn and Hodgkinson [6] in the middle 1800's. As is often the case today, the research was motivated by, and carried out in conjunction with, one of the most ambitious engineering endeavors of the day.

The adoption of wrought iron in civil engineering construction and the vigorous building programs of the then new railroads soon combined to bring to light heretofore unrecognized difficulties. Prime amongst these was the problem of plate stability. This came into prominence during the design of the Britannia and Conway tubular bridges. The dual requirement that the bridges should not hinder navigation but possess the rigidity necessary for a railway led to the novel idea that the bridges be constructed as large tubes through which the trains would travel.

William Fairbairn, one of the best known English engineers and experimentalists of his time, was employed to assist in the design and construction. Fairbairn began his career as a mechanical engineering apprentice and from this undoubtedly acquired the craft; but academically he was essentially self-educated in accordance with the British tradition of the time. In his day English engineers put little credence in mathematics, paid no attention to it in their education and depended primarily upon practical experience and experiment. Today, of course, engineers the world over are more deeply engrossed in the mathematical treatment and in this are aided by powerful computers; nevertheless experimentation still plays a vital role.

In order to determine the proper cross-section for the bridge.



Fairbairn had model tubes of square, circular and elliptical shape manufactured from sheet iron. The various and sundry joints were made by riveting. The models were supported at each end by timbers shaped to their cross-section. In addition, close fitting wooden plugs were inserted at each end. The models were between 17 and 32 feet in length and from one to two feet in diameter or lateral dimension. Each tube had a locally reinforced hole located on the center of the chord and span. An eight inch square wooden saddle or cushion was placed inside the tube concentric with the hole. This was used as a loading pad. Dead weight loading was made using carefully weighed pigs of iron and a loading platform. A lever and screw-jack between the loading platform and the suspension link supported the loading platform while the weights were applied. This screw-jack and lever system afforded a means of limiting the tube's deflection as well as removing the weight. Two views of the apparatus are shown in Figure 1.

Fairbairn's observations in these first experiments are best stated in his own words.

Some curious and interesting phenomena presented themselves in these experiments - many of them are anomalous to our pre-conceived notions of the strength of materials, and totally different to anything yet exhibited in any previous research. It has invariably been observed, that in almost every experiment the tubes gave evidence of weakness in their powers of resistance on the top side, to the forces tending to crush them.

As soon as he had discovered the phenomenon of plate instability, Fairbairn called upon his friend, Eaton Hodgkinson, for advice. Like Fairbairn, Hodgkinson was an eminent engineer and experimentalist. However, due to his mathematical education under the famous scientist

APPARATUS USED FOR THE EXPERIMENTS, WITH THE SHEET IRON TUBES & BEAMS.

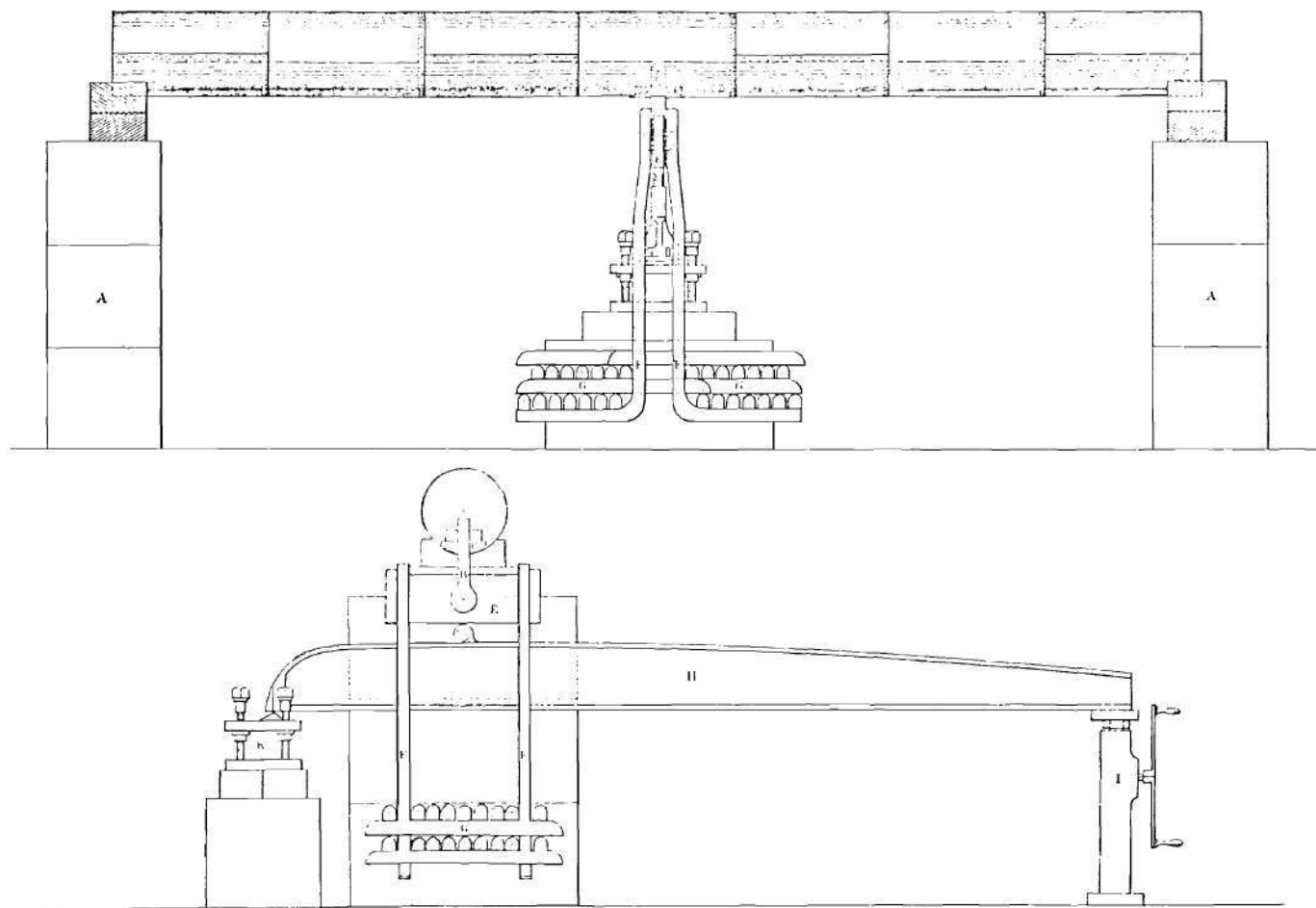


Figure 1. Test Apparatus Used by William Fairbairn (after Clark).

John Dalton, Hodgkinson possessed greater theoretical knowledge. After reviewing the problem, Hodgkinson expressed his opinion as follows:

It appeared evident to me, however, that any conclusions deduced from received principles, with respect to the strength of thin tubes, could only be approximations; for these tubes usually give way by the top or compressed side becoming wrinkled, and unable to offer resistance, long before the parts subjected to tension are strained to the utmost they would bear. To ascertain how far this defect which had not been contemplated in the theory would affect the truth of computations on the strength of the tubes ...

As a consequence of the initial observations, Fairbairn began a second series of tests to overcome the difficulties. With this aim, he conducted bending tests on a large iron model of 75 foot span and two and one-half by four and one-half feet rectangular cross-section. The model's compression side was stiffened by a cellular structure and its tension side was progressively stiffened until equal strength was achieved in compression and tension. At the same time, vertical stiffeners were added to the side plates to prevent wave formation. This also marked the first instance in the study of structural instability that the stiffness of a test specimen was successively varied and the effect noted.

In 1847, Fairbairn [7] conducted what well may have been the first caisson type tests of plates in compression. He used a rectangular cast iron tube (part of the bridge's compression side cellular structure) which was eight feet high by one and one-half feet square. It was fabricated by riveting half-inch thick plates to right angles. He used a large hydraulic press to axially compress the tube and observed "buckling" in the tube's walls.



During the same period, Hodgkinson [8] was conducting experiments on the strength of iron plates and tubes in compression. The plate specimens were 10 feet in length and from one-half to one and one-half inches thick with lateral dimensions of one to three inches. Hence, they really could all be classified as wide columns. These specimens were tested with flat ends and without side support. The axial compression was applied by two parallel horizontal plattens actuated by a system of levers. In this manner, Hodgkinson also compressed a series of 10 feet long tubes. These tubes had cross sections which were circular (one and one-half to six inches in diameter), square (four and eight inches on a side) and rectangular (eight by four inches).

In addition to the contributions of Fairbairn and Hodgkinson, the design of the Britannia Bridge also elicited the constructive criticism of Jourawski [6]. In his discussion, Jourawski made the significant proposal that test models should be made from materials with low moduli of elasticity since this would facilitate measurement of the elastic deformations. He constructed models of thick paper reinforced by cardboard stiffeners and, using these models, was able to show that inclined side stiffeners would have been more effective than the vertical side stiffeners used on the bridge. Today Jourawski's concept is widely used and much qualitative and quantitative work is based on it.

Following this pioneering work of Fairbairn, Hodgkinson and Jourawski, the potential of thin metal structures for ships became recognized. Thus, Bryan [9] in 1891, gave the first theoretical treatment

of the buckling of a simply supported, axially compressed rectangular plate in a paper entitled "On the Stability of a Plane Plate under Thrusts in Its Own Plane, with Applications to the 'Buckling' of the Sides of a Ship." This theoretical work was not matched by any substantial experimental endeavor. Indeed, it was not until a metal ship experienced difficulties that structural test work was undertaken by marine engineers. In 1905, the loss of the HMS Cobra, by suspected hull failure, caused Sir John Biles to perform tests [10] on her sister ship, HMS Wolf. His tests sparked a generous degree of controversy.

#### The Advent of Aviation

The introduction of metal construction in aviation provided the impetus for extensive programs of plate stability experimentation. The stringent requirement for the best strength-to-weight ratio could only be satisfied by clearly delineating experimentally the boundaries of stability for flat and curved metal panels.

Among the earliest experiments were those performed in 1930 by Schuman and Back [11] of the Bureau of Standards. Their test specimens were flat, unstiffened rectangular plates of either aluminum, stainless steel, nickel or monel metal. All the specimens were 24 inches in length, but their widths and thicknesses were varied. The loaded edges were milled flat in order to obtain a uniform distribution of load. Vertical bars with 45° V-grooves laterally restrained the unloaded edges. The buckle form was determined from measurements made with a dial micrometer attached to the test fixture. The apparatus is

shown in Figure 2.

Newell [12,13] reported work carried out between 1930 and 1932 by graduate students either at the Massachusetts Institute of Technology or at Stanford University. These studies covered the gamut of plate compression tests. In some of the tests the V-groove method of Schuman and Back [11] was used, but in the majority of cases crippling of the specimen's edge was prevented by use of a stiffener. This method gained wide spread use later.

In 1933, Sechler [14,15] at the California Institute of Technology undertook the experimental verification of von Karman's effective width concept. During the same year, Cox [16], of the National Physical Laboratory, conducted a series of tests designed to duplicate the lateral edge fixity encountered when a flat panel is riveted to a stiffener at its edges. To this end, his apparatus consisted of two square section columns mounted on a base plate. Each column had horizontal grooves, at a one inch vertical spacing, into which stirrups fitted. A rectangular notch in the side of each stirrup aligned it to the faces of the column. Once the stirrups were inserted, a cover plate clamped them against the column to prevent lateral motion. Each stirrup, a one inch wide strip of 16 S.W.G. duralumin, projected two inches from its column and terminated in a vertical face  $15/16$  inch high. The test specimen was bolted at its lateral edges to these vertical faces. Along the top and bottom loaded edges, the specimen was clamped between angles. This apparatus was noteworthy since it was considerably more elaborate than any previously used in plate testing.

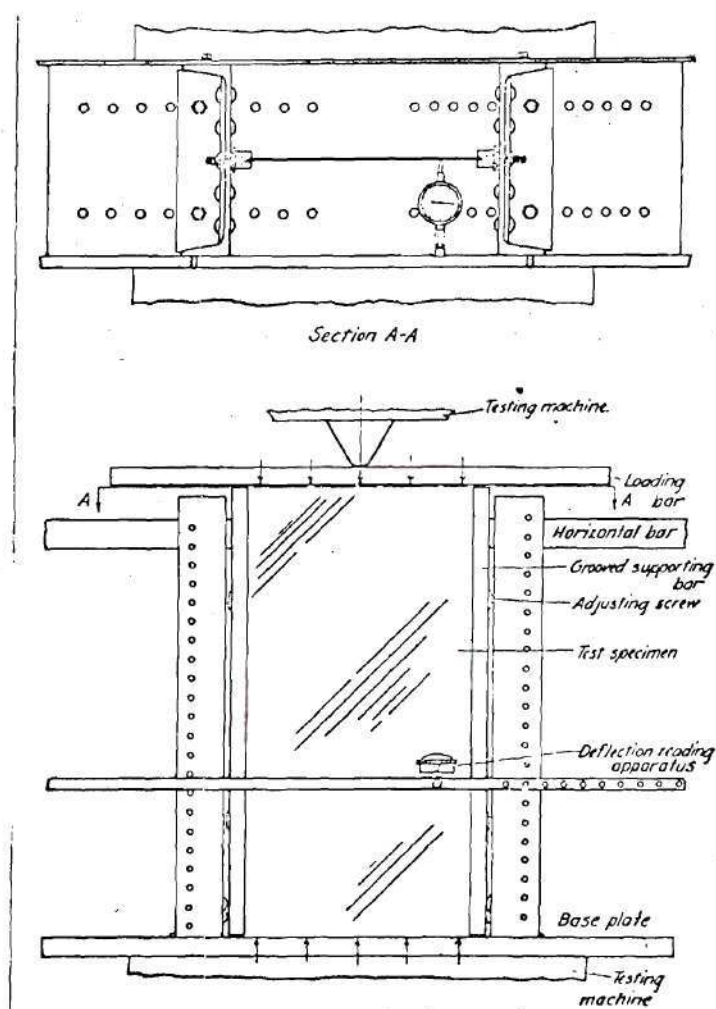


Figure 2. Test Fixture Used by Schuman and Back  
(after Schuman and Back).



Furthermore, it was designed specifically to provide such lateral restraint, in the plane of the sheet, as would actually occur in service.

The effective width of buckled sheets which had attracted the attention of von Karman and Sechler in 1933 was further studied experimentally, in 1936, by Lahde and Wagner [17]. Their tests were made on brass sheet clamped between angles on all edges (uniaxially compressed). The work was translated from the German and published by the National Advisory Committee on Aeronautics.

From the onset, it was realized that stringer flexural rigidity influenced the critical load level of panels. In 1936, Gerard and Dickens [18] raised the question of the degree to which the reinforcement torsional stiffness influenced the issue. To clarify this point, they tested curved aluminum panels stiffened by two tubular stringers. The panels were tested in longitudinal compression with the loaded edges cast in low melting point alloy and the lateral edges held between wooden clamps. They concluded that the tubular stringers were 15 percent more efficient on a strength-to-weight basis than extruded angle stiffeners.

By this time interests in structural stability were broadening. In Germany Ebner [19], who had also an interest in shell stability, attempted to correlate his studies of plates and his work on shells. His endeavor was not too successful. He succinctly stated the problem thus: "One difficulty, however, encountered in curved-panel tests is the correct choice of support conditions...".

A step forward in achieving simple support conditions was made in 1937 when Wenzek [20] arranged three outwardly curved panels in the form of a triangle. He loaded the resulting caisson in axial compression and thus, tested simultaneously three simply supported panels. Sechler [15] also alluded to Donnell using a four-sided Caisson with two inwardly and two outwardly curved panels. In 1939, Grady [21] tested flat panels with corrugated reinforcement using the caisson technique.

In 1939, Ramberg, McPherson and Levy [22] adopted a similar technique in their compression tests on a specimen of semi-monocoque construction. Their box specimen which had both stringers and ribs was fabricated of aluminum alloy. It was 95 inches in length and had a rectangular cross section of 24 inches by 10 inches. Fiscal limitations restricted the test program to a single specimen. Hence, care was taken to avoid a premature failure. The ends of the box were cast in plaster of Paris while a small compressive load was applied. The load distribution was checked by means of Tuckerman optical strain gages attached to the stringers and the corner posts.

In the same year, they [23] also investigated experimentally the deformation and effective width of axially compressed stringer stiffened flat panels. In this study, the emphasis was placed on carefully controlled test conditions rather than on the number of specimens tested. Hence, only two specimens of different sheet thickness were tested. The loading arrangement was designed to ensure a uniform stress distribution below the buckling load. This entailed casting the

loaded edges of the specimen in Wood's metal on faceplates. These faceplates which were attached to the upper and lower loading blocks rested in cylindrical bearings located in the centroidal plane of the specimen. The blocks were loaded through knife edges normal to the plane of the specimen. The lateral edges of the specimen were supported by two pairs of bars designed to approach the condition of simple support. The bars at each edge were separated by spacers whose thickness equaled that of the sheet. The two pairs of bars were held a constant distance apart by spreader rods. This allowed a small clearance between the sheet and the spacers. In this way, expansion of the sheet during compression was permitted.

Strain distributions were determined using Tuckerman optical strain gages on both the sheet and stringers. Pointers were also attached to the Z-section stringers of the specimens and photographs taken during the tests. The photographs were examined under a travelling microscope to determine displacements from which twists could be calculated. These deformations were analyzed using a Southwell plot where the deformation,  $\delta$ , is plotted versus the ratio of deformation to load,  $\delta/P$ . Excellent agreement was found between the results of this analysis and the instability load level. When the bending deformation measured by the Tuckerman gages was plotted, the agreement was not as good. In this instance, there were two approximately straight-line portions, one with a reciprocal of slope 34 percent higher than the ultimate load and the other with a reciprocal of slope equal to the ultimate load. The reciprocal of



slope yielding the correct ultimate load was obtained from measurements made close to failure. As a result, the authors made an astute observation, namely that :

The Southwell method must, therefore, be used with caution; a sufficiently large number of observed deformations must be plotted to establish the existence of a linear relation between  $\delta$  and  $\delta/P$  over a large range of deformations.

In addition to the measurements of deformations derived from the pointers and Tuckerman gages, plaster of Paris casts were made of the sheet contours after buckling. From these casts, shapes and depths of the buckles were determined.

The aim in the second part [24] of this research program was to determine the influence of fasteners. The effects of various types and spacings of fasteners on the strength of panels and the nature of buckling between fasteners and between stringers were therefore the subject of the study published in 1942. The panels used were stiffened by Z-section stringers attached by either brazier-head rivets, round-head rivets or spot welds. Tuckerman optical strain gages were positioned on both flanges of the stringers but not on the sheet. The lateral deflection of the sheet, midway between fasteners along a stringer, was measured by means of a dial gage mounted on a special bar located on the fasteners. This same device was also used to measure the lateral deflection of the sheet midway between stringers. Since the bar was located on fasteners, it was not necessarily at the crest of the buckle between stringers.

On all panels, the loaded edges were ground flat and cast in

Wood's metal. For panels with fastener spacing equal to or less than the stringer spacing, the lateral edge bars of the earlier test were again used. When the fastener spacing exceeded the stringer spacing, these bars were modified to provide a localized support along the line of fasteners. Both methods of edge restraint are shown in Figure 3.

A new method to approximate simple support along the lateral edges of a panel was utilized by Dickinson and Fischel [25] in 1939 and by Dunn [26] in early 1940. The method was to insert the lateral edges of the specimen into slotted tubes. The slots were sized so that they would permit the necessary translation and rotation of the panel edges. Dickinson and Fischel's sole contribution was the use of this method of edge restraint. Dunn gave a thorough description of both his tests and specimens. He also presented a theoretical study resulting in a procedure for calculating the buckling stress of a plate whose lateral edges were elastically supported and whose loaded edges were simply supported.

Dunn tested 183 flat aluminum panels each stiffened by three equidistant bulb-angle stringers. Twelve stringer sections and three sheet thicknesses were used in these specimens. The loaded edges were milled flat and the load distribution checked by tensiometers. Nonparallel motion of the movable head relative to the fixed base of the test machine necessitated shimming the face plates between the specimen and the heads to obtain uniform load distribution.

The wave form of the buckled sheet was traced and recorded

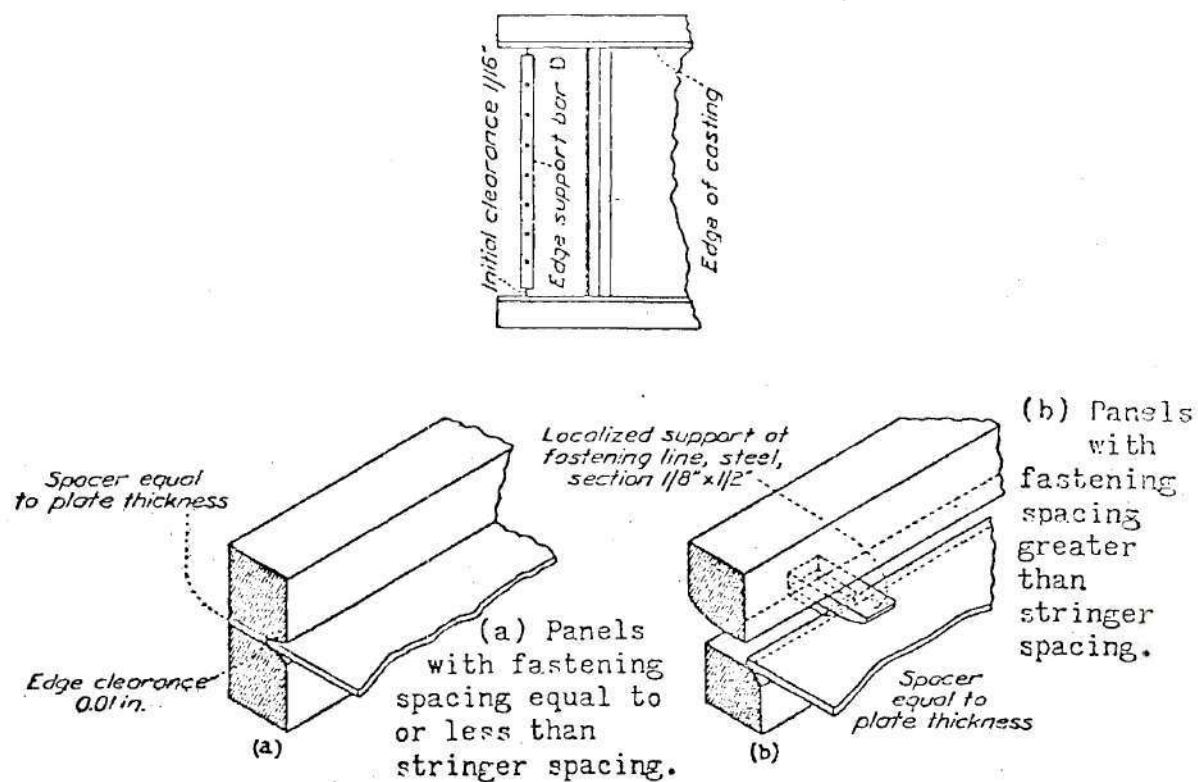


Figure 3. Edge Supports Used by Ramberg, McPherson and Levy (after Ramberg, McPherson and Levy).

with a special apparatus. This consisted of a rack and roller device mounted on a carriage which moved on a vertical column. The roller followed the surface contour of the sheet and, through an intermediate amplifying gear train, operated a pen which traced the wave profile on a recording drum. A light spring ensured contact between the roller and the specimens.

Dunn observed that plotting  $(f_o/\lambda_o)^2$ , where  $f_o$  is the buckle amplitude and  $\lambda_o$  is the buckle width, as a function of the axial stress,  $P/A$ , yielded a straight line. He found that this straight line intercepted the  $P/A$  axis at the value of the buckling stress. Thus, Dunn was the first to make use of the  $P - \delta^2$  law to interpret experimental data and by doing so, made an important contribution to the study of structural stability. Due to its elegance, the proof given by Dunn is repeated here.

Since  $P$ , the applied load, is independent of the direction in which the sheet buckles, we can write

$$P = \text{an even function of } f_o/\lambda_o \quad (1)$$

Then, by a Taylor's expansion

$$P = P_c + P''(f_o/\lambda_o)^2/2! + P'''(f_o/\lambda_o)^4/4! + \dots \quad (2)$$

putting

$$(f_o/\lambda_o)^2 = u \quad (3)$$

We have

$$P = P_c + A_1 u + B_1 u^2 + \dots \quad (4)$$

Plotting  $u$  as a function of  $P$ , the resulting curve will be very close to a straight line for small values of  $u$  and  $P_c$  will correspond to the buckling load. The proof's generality is unaltered if  $P$  is divided by the cross sectional area  $A$  of the panel.

The influence of conditions at the loaded edges was first investigated by Holt [27], in 1941. His study was made on aluminum panels stiffened by top hat stringers. Three different methods of applying end restraint to the panels were utilized. The first and second specimens had their loading plates supported on knife edges. On the first specimen, the centroidal axis of the specimen passed through the knife edges. However, the second specimen was shifted on the platens until sheet and stringer strain measurements indicated a nearly uniform distribution. The third and the fourth, which was twice the length of the third, were tested as flat ended columns, i.e., the loaded edges were machined flat and parallel. The fifth specimen had its loaded edges rough cut rather than machined. When it was compared to the third specimen, which was the same length, the ultimate strength was only a few percent less. Hence, the difference between the end conditions of the two specimens did not appear significant. The sixth and seventh specimens, with machined ends, were supported laterally at intermediate points by one inch



square steel bars attached across their width. The ends of these bars were connected by rods to a steel framework made from I-beams attached to the test machine. The rod connections were designed to minimize rotational moment due to relative motion between the specimen and framework. Consequently, they provided only a minimal restraint to the specimen's rotation. The sixth specimen was divided into three 12 inch long bays by these lateral bars. Similarly, the longer seventh specimen was divided by the bars into five 12 inch bays. These seven specimens were tested in a hydraulic-type test machine and an eighth specimen, identical to the third, was tested in a screwjack-type machine.

As a result of his experiments, Holt concluded that:

(a) For the slenderness ratios used, specimen length had an insignificant effect on the ultimate compressive load.

(b) The precision with which the specimen ends were machined flat and the care exercised in alignment of the specimen had no consistent effect on the ultimate load, although they increased the uniformity of the stress distribution in the elastic range.

(c) Whether short specimens were tested in a hydraulic-type test machine or a screwjack-type test machine seemed to make no difference in the ultimate load.

In 1941, Cox and Clenshaw [28] of the N.P.L. published the results of work which they had conducted in 1934 on curved plates under axial compression. The test apparatus for these curved plates was more elaborate than the one developed previously by Cox [16] for

flat plates. The fixture was constructed in such a manner as to accommodate several widths and thicknesses of specimen and is shown in Figure 4. The top and bottom edges of the plates were attached to the upper and lower platens by appropriately formed angles. The lateral edges of the plates were guided by a series of slotted strips which were supported between vertical columns. The slotted plates were interleaved with rubber blocks thus forming a comb arrangement. These strips together with the pair of supporting columns formed one unit that could be rotated front to back or top to bottom. The strips in the comb had two slots cut in each of the two sides between the columns. These cuts were made slightly off the center line so that whatever possible positions of the comb (rightside up and side A inward, upside down and side A inward, etc.) would bring one of the four different widths of slot into alignment with the upper and lower angles. Thus, four different thicknesses of plate could be tested. In addition, since the comb columns rested in cylindrical cups attached to the lower platten, cup positions were provided for testing plates 4, 6, 8, 10 and 12 inches in width. At their tops, the comb columns were braced laterally by bars connecting the tops of corresponding columns. This apparatus was the first designed to simulate clamped edges and Cox stated that it performed quite satisfactorily.

#### Research During World War II

As a result of the global conflict, interest in plates intensified.

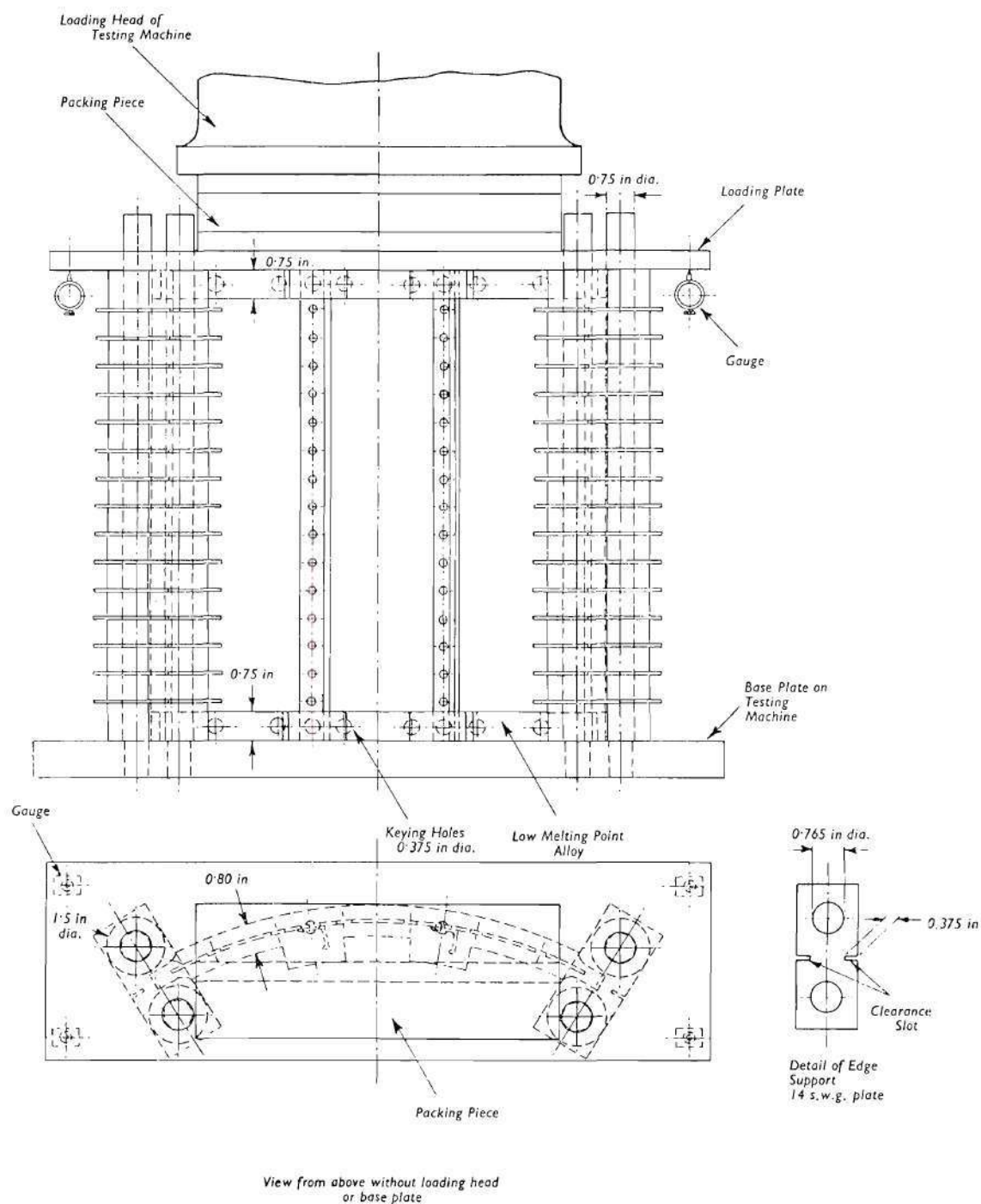


Figure 4. Test Fixture Used by Cox  
(after Cox and Clenshaw).

Most of the research conducted during World War II was not published in the open literature until well after the War's conclusion. Much of the work was ad hoc, being conducted for a specific project, but some valuable basic research was also performed.

The volume of work generated during the period was such that strict chronological discussion becomes unwieldy. Thus, for the sake of convenience, the various studies of import are grouped according to character rather than date. Categories inevitably lead to some difficulties since for every item which can be described as black or white, there is always one which is gray. But, broadly speaking, the three categories to be dealt with are:

- (1) Flat, stiffened aluminum panels tested flat ended (loaded edges machined flat), with stringers at the lateral edges, under axial compression.
- (2) Curved aluminum panels, with and without stiffeners, under axial compression.
- (3) Caissons or airfoil segments, with and without normal pressure, under axial compression.

The first work in category one was performed by Niles [29] in 1943. He tested 51 panels which included all possible combinations of two lengths, four stiffener spacings and four stiffener cross sections. In addition to measuring the panel end shortening by dial gages, he also glued pointers to the stiffeners to measure any twisting tendency and took photographs at failure. This study was aimed at obtaining copious data on the influence of each stiffener type on



the buckling of the plate. Rossman, Bartone and Dobrowski [30] tested in excess of 475 panels with Z-section stringers in 1944. They published no information as to their test procedures or measurement techniques. Since they listed the critical stresses, it would appear that they were interested only in the effect of panel dimensions on the compressive strength. Kotanchik, Weinberger, Zender and Neff [31], in 1944, tested 247 panels with Z-section stringers and 304 panels with hat-section stringers. In these tests the height, thickness and spacing of the stiffeners, the thickness of the sheet, and the length of the panel were systematically varied to ascertain the effects of these dimensional changes on the compressive strength. Their conclusion was that the average stress when sheet buckling occurs is in good agreement with the theoretical predictions. In contrast to the work of Rossman, Bartone and Dabrowski [30] just cited, the specimens in these tests were instrumented to acquire strain data. Strains in both the sheet and the stringers were measured at the midsection of the specimens using Tuckerman optical strain gages and wire resistance-type strain gages. This is one of the earliest instances of electrical resistance wire strain gages being used in panel testing. The study was extended by Weinberger, Rossman and Fisher [32] to determine the effect of using a different aluminum alloy for the panel sheet. Accordingly, 48 panels with hat-section stringers were tested and compared with a geometrically identical group of panels examined in the previous study. The testing procedures were, of course, almost unchanged; the only variation



was that the electrical resistance-type strain gages were omitted. Dial gages were again used to measure end shortening. Dow and Hickman [33] conducted a series of panel tests to determine the relationship between panel strength and the diameter of the rivets used for stringer attachment. Their specimens were constructed with lengths always less than half their widths and were stiffened by stringers of an inverted V cross section. In 1945, Dow and Hickman [34] continued their investigation into the effect of variation in rivet diameter and pitch. The 140 specimens used in the later tests were stiffened by closely spaced Z-section stringers and had lengths which exceeded their widths. The specimens were designed to fail by local buckling and only failing loads were recorded. In the same year, Schutte [35] published charts for the minimum-weight design of flat aluminum alloy panels with Z-section stringers. For these charts, he utilized the test data which had been reported by Rossman, Bartone and Dobrowski [30] as well as the results from tests which they completed after publication of their report. These additional tests were conducted in the same manner and on the same types of specimens as the original tests.

In late 1941, Lundquist [36] conducted research in the second category. The specimens, seven curved and one flat, were of aluminum alloy with two angle stiffeners, back-to-back at each lateral edge. These angles were used to preclude edge failure and were sized such that buckling would occur in the sheet. Each specimen was instrumented with three pairs of Tuckerman optical

strain gages applied back-to-back across its mid-section. No deflection measurements were taken. Lundquist used the strain measurements to determine the buckling load for the flat and the least curved specimens. For this purpose he had developed [37] a special plot - the Lundquist Plot - from the Southwell Plot. This apparently was the first time that such a method was applied to experimental data obtained from the compression of a slightly curved panel. He did not use the method for the highly curved specimens since he was only interested in knowing the critical load and the highly curved specimens failed at a clearly defined load with "a loud report." Lundquist concluded from his tests that for a curved sheet the critical stress between stiffeners was equal to the larger of either:

- (a) The critical compressive stress for an unstiffened circular cylinder of the same radius-thickness ratio, or
- (b) The critical compressive stress for the same sheet when flat.

Holt [38] in 1943, sought to obtain specimens representative of typical aircraft construction. To do this, he had six specimens of various sizes cut from an actual aircraft wing. The effects of curvature were studied by making successive tests on each specimen using templates of various radii. The specimens were elastically sprung to fit the templates clamped to the platens of the test machine. Longitudinal strains were measured along the transverse center line on both sides of the specimens by using Huggenberger

Tensiometers. After the curvature tests, the specimens were tested to failure as flat panels. Holt concluded:

The critical buckling strain of stiffened curved sheet in the elastic range varies linearly with the ratio of the thickness of sheet to the radius of curvature and can be computed with reasonable accuracy by the equation given by Wenzek.

$$\epsilon_c = (\sigma_c/E) = 5\left(\frac{t}{b}\right)^2 + 0.3 \left(\frac{t}{R}\right) \quad (5)$$

where  $\sigma_c$  = buckling stress  
 $\epsilon_c$  = buckling strain  
 $t$  = thickness of the sheet  
 $b$  = width of the sheet  
 $R$  = radius of curvature

The work on curved panels, begun by Lundquist [36], was continued by Crate and Levin [39] in 1943. Their 53 specimens had the same aspect ratio as those of Lundquist and the angle stiffeners at the lateral edges were identical. However, Crate and Levin used four additional thicknesses and altered the positions of the Tuckerman gages. They arranged their eight pairs of back-to-back gages in two staggered vertical columns about the panel's upper two quadrants. Due primarily to the quantity of specimens used, these became for a number of years some of the most frequently referenced curved panel tests in the United States. Crate and Levin reached precisely the same conclusions as Lundquist in regards to the critical stress of

curved panels.

The following year, Ramberg, Levy and Fienup [40] tested 21 curved panels with an aspect ratio of three-fourths. The specimens were stiffened by Z-section stringers to which Tuckerman strain gages were attached on the upper and lower flanges. Some SR-4 wire strain gages were also attached to the stringer webs. The loaded edges of these specimens were ground flat and in some instances also cast in Wood's metal, but no difference in behavior at the edges was observed for the two procedures. The lateral edges of the specimens were restrained by the same edge guides as were used in the tests of references [23] and [24]. The effective widths determined from the tests were compared with those derived from various theoretical and empirical formulae. The critical buckling strains for the specimens were also determined and compared with the pertinent theoretical predictions.

McPherson, Fienup and Zibritosky [41] extended this study with six panels whose aspect ratio was one-half. The same edge conditions were again utilized, viz., the loaded edges were ground flat and the lateral edges were held in edge guides. Tuckerman optical strain gages and SR-4 wire strain gages were also attached to the Z-section stringers, as in the previous study. The increased developed width of these six specimens was found to have no significant effect on the strain for buckling of the sheet between stringers, the strain for buckling between rivets, the load carried per sheet bay, or the



stress at failure when compared to the previous specimens. It did, however, "reduce the critical strain for buckling of the panel as a whole between the edge guides."

In the third category, Rafel [42], in 1942, tested two caissons each composed of two outwardly curved panels joined by edge flanges. These caissons, which differed only in the number and spacing of internal ribs, were tested under axial compression with internal pressure also applied. In this study of the effect of normal pressure on the buckling load, Rafel was the first investigator to use normal force and axial compression in a curved panel stability problem. Rafel found as a result of his tests that "an outward acting normal pressure very appreciably raises the critical compressive stress for unstiffened curved sheet."

The desire to attain higher aircraft speeds had made designers acutely aware of the drag due to surface waviness with thin skins. Thus, in early 1943, Jacobs, Lundquist, Davidson and Houbolt [43] tested a low-drag airfoil specimen under axial compression and examined the surface waviness produced at various load levels. This specimen was stiffened by Z-section stringers. A second specimen stiffened with hat-section stringers was tested by Davidson, Houbolt, Rafel and Rossman [44]. Their tests were more elaborate than those of reference [43] since a wind tunnel test to determine the drag characteristics was conducted following each compression test. However, they were forced to conclude that two specimens gave insufficient evidence on which to base a decision as to which construction type to use.



Interest in the subject continued and at the same time plastic and bonded materials came to the forefront as potential primary structure materials. Schutte, Rafel and Dobrowski [45] thus initiated a study which covered both aspects. They tested six curved paper-base plastic caissons subject to axial compression and internal pressure. The specimens were internally stiffened by spruce stringers and had spruce end spars joining the two outwardly curved panels. A number of electrical resistance strain gages were attached to the inner and outer surface of each panel. Little meaningful data was obtained from the strain gages since the failure mode was a separation of the skin from the stringers. Data on the lateral deflection of the skin was obtained, however, with dial gages.

In 1945 Rafel and Sandlin [46] extended the work on the influence of internal pressure on the stability of compression surfaces. They tested 20 caisson specimens under combined loading. These two-panel caissons were combinations of five different radii of curvature and four different rib spacings. The specimens had several resistance-type strain gage rosettes attached back-to-back in the center bay of one side. In these tests, as in the previous ones, a mercury manometer was used to measure the internal pressure. Rafel's earlier conclusion that normal pressure appreciably raised the critical compressive stress was confirmed. The conclusion reached was that the greater the ratio of radius of curvature to thickness of sheet, the greater the percentage increase in critical compressive

stress with increasing normal pressure.

Several other panel experiments carried out at the N.A.C.A. during World War II should also be noted. In two of these studies the engineers had a peripheral interest in the effect of ribs on the buckling of uniaxially compressed panels. Lundquist and Schwartz [47] (1942) were interested in the design of the rib-chord members of truss-type ribs. Consequently, they constructed test panels to obtain data on the effect of variations in the distance between the truss-to-panel attachments points. They assumed that the chord members of each rib structure possessed sufficient bending stiffness to be equivalent to a rigid member between their end supports. Thus, they felt buckling would develop between the ribs, however, this was not substantiated by the experiments. The failures were not due to local buckling between the ribs but rather to a general instability which involved the ribs as well as the skin. Hence, the actual strength of the specimens was less than the design strength.

In 1943, Scott and Weber [48] designed a fixture to subject rib stiffened panels to compression and combined compression and shear. In this device, offset knife edges applied load at the top and bottom of each specimen. Lateral rollers were used to supply the reacting forces. All panels had vertical stiffeners along their lateral edges and from one to three rectangular cross section ribs of various depths. A total of 21 specimens were tested in pure compression.

The experimental results were compared with those of a

theoretical study by Timoshenko [49]. However, the test fixture did not duplicate the boundary conditions assumed by Timoshenko. Moreover, the ribs being attached on only one face of the panel also differed from those prescribed in Timoshenko's analysis. Hence, the difference between theory and experiment (the values from the experiments were higher) was considered explicable. The test apparatus is shown, in a combined loading configuration, in Figure 5.

A series of tests in direct support of the development of the XB-36 long range bomber were conducted by Weinberger, Sperry and Dobrowski [50] in 1944. A number of electrical resistance wire strain gages were used in these tests of 63 flat panels stiffened by corrugated sheets. In the same year, an investigation was conducted by Zender, Schuette and Weinberger [51] into the compressive strength of stiffened panels fabricated from a composite of plastic-bonded glass cloth and canvas layers. The U-shaped stiffeners were molded from the same material and glued to the sheet. Strain gages were attached on both sides of the specimens between and on the stiffeners. The critical compressive stress for the sheet between and on the stiffeners was determined from the strain gage readings by means of the Lundquist plot [ ].

Due to the exigency of World War II, plywood resurged as a material for airplane structures. The most notable example of its use was the DeHavilland Mosquito whose illustrious military record gives striking testimony to the many admirable qualities of the material. The Forest Products Laboratory of the U.S.D.A. Forest Service

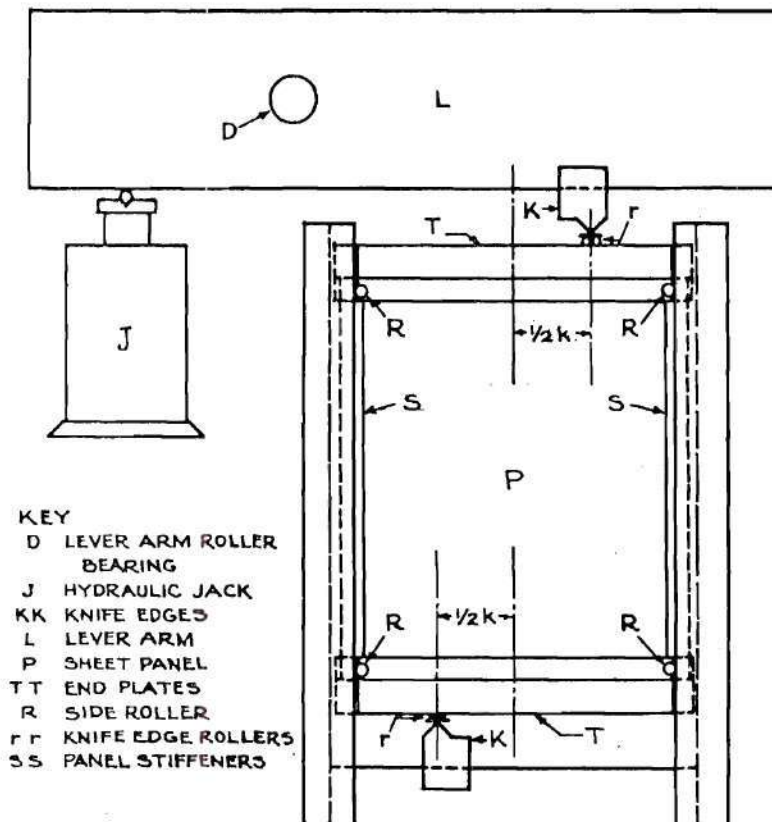


Figure 5. Sketch Showing Method of Loading Sheet Panel in Compression or Combined Compression and Shear (after Scott and Weber).



conducted, during 1944, experiments on the buckling of thin, curved plywood plates subjected to axial compression [52]. In the majority of cases the plates had a five and one-half inch radius of curvature and were of widths from one-half to four times this radius and lengths from one to six times it. Various alignments of the face plies to the direction of compressive load were utilized in these specimens. The lateral edges of the specimens were restrained within metal angle pieces. These vertical angles were adjusted until there was a uniform clearance of .002 inch along the length of the panel. Thus, the lateral edges of the panel were guided but were not clamped. This guiding restraint was only in the radial direction. Before testing, the upper and lower loaded edges of the specimens were finished on a planer. The upper edge bore directly on the test machine loading head while the lower edge bore on a plate mounted on a spherical bearing attached to the test machine platten.

To investigate the effect of the guide angles on the buckling of the panels, cylinders of the same radius and material were tested in axial compression with internal "spiders" to restrain buckling. These "spiders" were arranged so that six or 12 equally spaced radial vanes were held against the inner wall of the cylinders, thereby preventing buckling along their vertical lines of contact. The vanes were one-fourth inch shorter than the cylinders so that they would not contact the loading head. Difficulty was experienced in adjusting the vanes of the spider to fit exactly the inside of the cylinders prior to the application of compression. Thus, occasionally the width of



the buckle was greater than the vane spacing. However, this may have been due to the buckles in adjacent bays causing local increase in the specimen's diameter which allowed a buckle to increase in width. Effectively, these "spiders" were discrete mandrels which prevented a catastrophic failure of the test specimens under axial compression. Hence, they can be considered as the precursors of the solid mandrels developed and used by Horton [53] and his associates for the testing of unstiffened metal cylinders under axial compression. Of course, mandrels had been used in the testing of pressure vessels also, but in such cases their primary function was to lessen the volume of fluid needed. This work appears to be the first in which a mandrel was used for tests under axial compression. The buckling stresses of the cylinders containing these internal mandrels were found to be about 2 percent higher than these of the corresponding curved panels. This is not surprising since the shell panel boundary conditions and the individual panel boundary conditions are somewhat different. This is explicable as the curved panel edges being able to rotate slightly and to move circumferentially while the cylinder's continuation of material over a restraining vane provides more of a fixed edge condition.

The De Havilland Mosquito also provided an impetus to the study of sandwich construction since its fuselage was fabricated in that fashion. To investigate the potential of this technique, Hoff and Mautner [54], in 1945, tested 51 flat rectangular sandwich panels in axial compression. Their test fixture was virtually identical to the

one used by Schuman and Back [11]. A flat steel plate with vertical adjustment bolts was placed between the test machine loading head and the panel's upper edge. By this means, the panel could be loaded uniformly along its upper edge even when this edge was not parallel to the loading head. Strains were obtained by using three pairs of back-to-back SR-4 strain gages attached to each panel. The bolts in the loading plate were adjusted until the strain gage readings indicated a uniform strain distribution at several values of axial compression.

As was the case with the N A C A , the National Research Council of Canada did not publish openly the results of a substantial number of tests on curved plates in compression conducted during World War II. The data was first published in 1945, and to gain wider dissemination, republished in 1947. In this study, Jackson and Hall [55] performed two series of tests comprised of 155 curved and 24 flat, unstiffened aluminum alloy plates. The two series of tests differed in the method used to impose the boundary conditions although both methods were intended to simulate clamped edges. Since the boundary restraints were different, the two series of tests will be discussed separately.

The first series of tests utilized a modified "comb" assembly, modelled after the one developed by Cox and Clenshaw [29], along the lateral edges while the loaded edges were cast in Woods' metal. This method, used for 103 curved and 15 flat plates, was deemed unsatisfactory for the following reasons:

(1) The width of the slot in the "comb" assembly had to accommodate the thickest specimen of a given nominal thickness and hence, the thinnest specimens of the same nominal thickness were permitted some free movement.

(2) The thin comb plates, spaced at half the assumed rivet pitch, did not provide adequate edge support and premature edge failure resulted.

(3) Casting the ends of the specimen in Wood's metal caused distortion due to thermal effect.

(4) Inadequate bearing strength at the corners of the specimens after buckling and variations in positioning of the specimens in the test machine also contributed to the problems.

During the second series of tests, comprised of 52 curved and nine flat plates, the "comb" assembly was altered by riveting "stiffener blocks" between pairs of "comb" plates at the edge by the specimen. Each of these "stiffener blocks" ended in a vertical tab to which the specimen was attached by a nut and bolt. This "revised comb" assembly was designed to provide the support of a stringer, albeit discontinuous, and to carry no axial load. The load edges were now clamped between blocks rather than cast in Wood's metal. The "revised comb" fixture is shown in Figure 6.

The initial deformation magnitude and the growth of the buckle pattern of each of the specimens was recorded photographically. A line filament lamp cast sharp shadows of an array of straight, one-half inch spaced, vertical wires onto the surface of the specimens.



# REVISED COMBS

## FOR PART II TESTS

SHOWING UPPER BRASS PLATE REMOVED

THESE COMBS ARE DESIGNED TO PROVIDE THE SUPPORT  
OF A STRINGER BUT ARE DISCONTINUOUS AND CARRY  
NO LOAD.

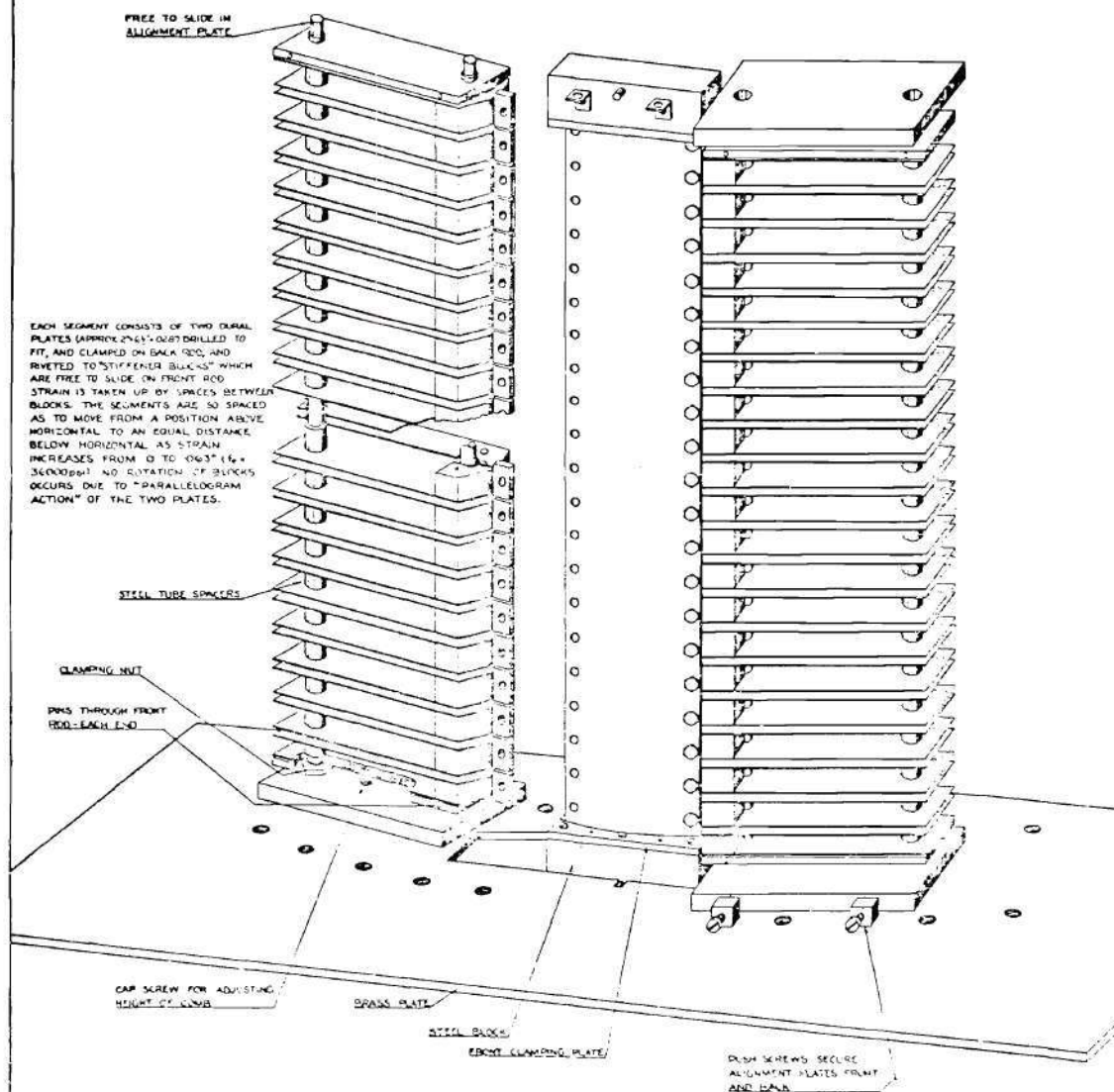


Figure 6. Revised Comb Assembly  
(after Jackson and Hall).

This shadow pattern was photographed and then contour diagrams and section profiles of the deformed specimens were derived from measurements of the photographs. According to Jackson and Hall, the chief advantages of this method were:

(1) The speed with which measurable records could be obtained during a test.

(2) The amount and accuracy of the information obtained from measuring the photographs.

The general testing arrangement, including the photographic equipment and the "combs" used in first series of tests, is shown in Figure 7.

Cox, Thurston and Coleman [56] again used the "comb" method of lateral restraint, in 1945, in the testing of seven flat panels stiffened with Z-section stringers. As was characteristic of much of the experimental research conducted during the war years, these tests were intended, in the words of Cox: ". . .to provide specific data for application to a particular design problem. . ." The test fixture and specimen preparation again proceeded along conventional lines. The upper and lower edges of the panels were cast to a depth of one inch in a low melting point alloy and then ground flat and parallel. Between the lower edge of the specimen and the test machine platten, there was a three-fourths inch thick flat plate to which the two pairs of vertical columns for the "comb" assembly were mounted. The slotted flat plates of the "comb" array were spaced along these columns by rubber washers. To preclude the possibility of premature failure due to local edge buckling of the panels, the "comb" plates were closely spaced;



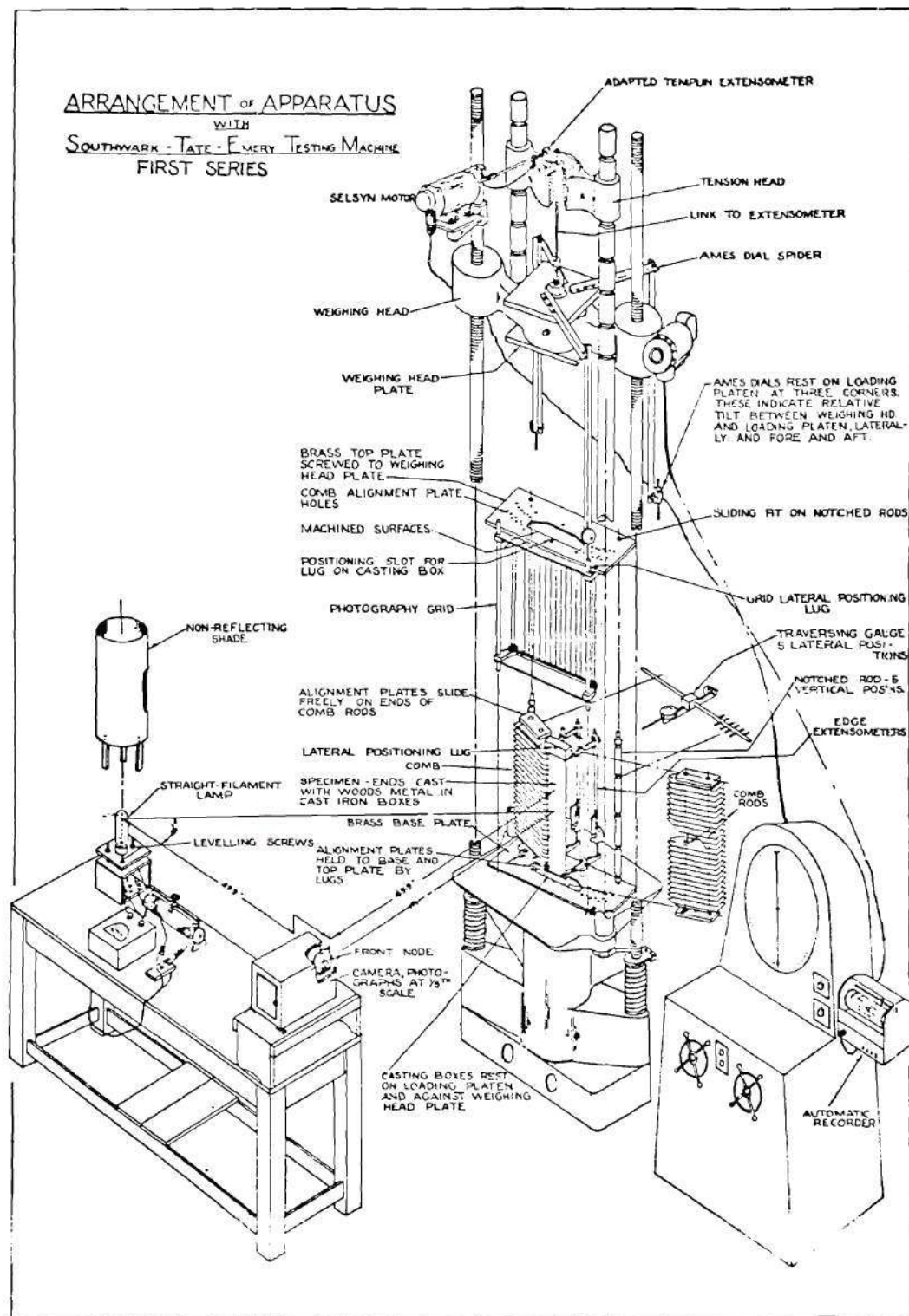


Figure 7. General Testing Arrangement  
(after Jackson and Hall).

in some cases, every one-half inch.

Since the panels were wide and short, 20 & 1/2 inches wide and 12 & 1/2 inches long, and the stringers were light in comparison to the sheet; special precautions were taken to ensure a uniform distribution of load across the width. This was accomplished by a "pyramidal" loading arrangement. A three-fourths inch thick plate and then a five inch deep steel I-beam were laid across the top of the panel. Two rollers, one inch in diameter, were placed on the I-beam with a spacing between them of five-ninths of the length of the beam. A second I-beam was placed on top of these two rollers and this beam was loaded at its center through a roller bearing. This arrangement resulted in a very uniform distribution of load across the width of the panels. However, care was necessary to prevent bending in a plane perpendicular to the panel's width due to the flexibility in that plane. Due to this characteristic of the loading system, the locations of the electrical resistance wire strain gages were selected to allow a complete survey of the strain distribution in the panels. There were six pairs of back-to-back gages located at three positions across the width near the panel mid-height. One pair of gages was attached to the sheet adjacent to a stringer and another pair on that stringer's web. The three positions selected were on the central and the two outer-most stringers.

In 1945 Welter [57] published the first of three papers detailing the results of experiments on curved aluminum alloy sheets in compression. This work was notable for its scope; a large number

of specimens was tested and a wide variety of factors which influence buckling were investigated. It was Welter's intent to develop both equipment and procedures which minimize the time required to accurately test a large number of specimens. Speed in testing was certainly achieved. Welter stated that a well-trained crew could test approximately 100 panels in 16 hours.

Initially, Welter had the loaded edges of his specimens cast in Wood's metal but he encountered the same difficulties as had Jackson and Hall [55]. He concurred in their conclusions that casting the ends released stresses and caused deflections of the plates due to the pouring temperature of the low-melting point alloy. It also caused uncertainties of load distribution at the ends of the plates. For these reasons, he resorted to having the loaded ends clamped between steel grips.

Welter rightly observed that "buckling is without doubt influenced by the stiffness of the longitudinal edges of the plates." To study this influence, he developed several different models of stringers which were fastened to the longitudinal edges of the plates. These consisted of small flat stringers, larger angle stringers and stringers with a free space representing rivet pitch. A series of stringers of variable length were also designed and used in the investigation of plate length effects. It was ascertained from these preliminary tests that edge stringers exerted a large influence on the buckling strength of panels. It is important to emphasize here that Welter's work was the first in which the influence of lateral edge



stiffness on the buckling load was noted. Welter decided in his subsequent investigations to use a far stiffer edge stringer than previously. The new stringers were flat bars which had teeth machined along their leading edge. The stringers were separated by "shims" whose thickness matched that of the plate and then clamped in such a fashion that the teeth gripped the edges of the panel. The free spaces between the teeth which simulated rivet pitch were varied from one inch to zero (continuous stringer) in one-fourth inch steps. The clamping depth, i. e., the teeth width, was varied from one inch to one-fourth inch again in one-fourth inch steps. These systematic tests were also the first made to determine the influence of the depth of the lateral edge constraints on the maximum buckling load and the ultimate load of panels. Welter found this parameter to have a "characteristic and pronounced" influence. His results are reproduced in Figure 8.

Welter also investigated the influence of panel length as well as the direction of rolling and imperfections in the parent sheet from which panel specimens were cut. The length of the panel was found to be important only when it was less than three-quarters of the width. The variation of buckling load due to the rolling direction was examined by loading plates whose edges were cut parallel, transverse to and at 45 degrees to the rolling direction. Plates loaded along the transverse direction were found to have higher (approximately 10 percent) average buckling loads and ultimate loads. This was felt to be due to the effect of cold-stretching. A survey of the load versus position in the parent sheet from which a plate was taken indicated

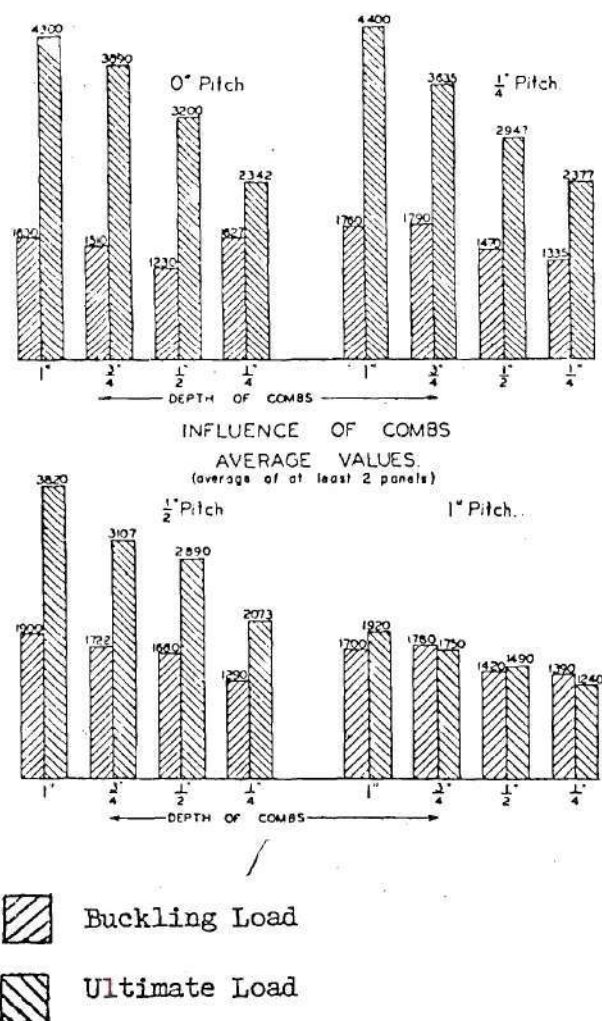


Figure 8. Influence of Combs on Buckling Loads (after Welter).



that the highest buckling loads were obtained for specimens taken from the middle of the sheet. Thus, it was concluded that the borders of the parent sheet were more susceptible to imperfections particularly with regard to thickness.

In his second paper, Welter [58] described tests on panels which had a 24 inch radius of curvature and a length of nine inches between the end grips. These panels were all constructed from .032 inch thick aluminum alloy sheet. In the first phase of this test series, Welter attempted to achieve a distribution of compressive stresses which would allow the ultimate load to be reached without buckling taking place. To do this, he made various patterns of slots and holes in the panels. Of the 30 panels so altered, some exhibited only a small difference of slope between the elastic range and permanent deformation in the load deflection diagram.

For the second phase of his tests Welter's concern was with the influence of the panel width and the width of the supported edges on the buckling and maximum loads. The results of these tests demonstrated that edge guides, one-half inch in width, could have an influence on a panel in which the depth which was greater than the width. The effect of variation in the clearance between the teeth of the edge guides and the panel edge was also examined. The highest buckling loads were achieved when the guides were adjusted to the exact thickness of the panels. Under these circumstances, the buckling loads were about 10 percent higher than when the guides were clamped tightly to the panels and about 15 percent higher than when there was

a clearance of .012 inch between the guides and the panels. As

Welter stated:

From the results of these tests it becomes obvious that the influence of the clearance of the guides is of material importance. Guides with a nonadjustable slot and a given tolerance do not seem to give the highest possible loads of the panels; it appears to be necessary to adjust the guides individually to each panel.

In his third paper, Welter [59] dealt with the effects of rolling the panels to smaller radii and then elastically spreading them to the test radius of 24 inches. The tighter radii used started at six inches and increased in steps of two inches. Also, flat panels and panels with a negative curvature of 24 inches were tested at a radius of 24 inches. After this operation was performed, both the buckling and ultimate loads were distinctly higher. In fact, there was almost a 100 percent increase in the buckling load over the panels which had not been rolled to a smaller radius.

Welter was the first investigator to impose initial indentations in his specimens in order to ascertain their influence on the buckling load. He hoped that the behavior of panels with known imperfections would shed light on the nature of imperfection sensitivity. A total of nine indentations were made with a 10 mm steel ball in the upper third of a panel. Various pressures were applied to the ball so that the indentations varied in depth from between one to three times the thickness, .032 inch, of the sheet. But in the words of Scotland's most famous poet, Robert Burns: [60]" . . .the best laid schemes o'mice and men gang aft agley." The panel buckled in its lower section

at a load which was not different from that which caused buckling in panels without indentations. Undaunted, Welter repeated the test on a panel with indentations made by a one inch diameter ball. Again, various pressures were applied to the ball so that the indentations were of different depths. Again, the buckles occurred in the area free of indentations --the middle of the panel, but the buckling load was comparatively higher for this ball size. A third panel had "three indentations of maximum effect" produced by three bullets from a .22 inch rifle. In Welter's words, ". . .buckling took place in the usual manner, not in the least influenced by this kind of deformation."

#### Post-War Research

The immediate postwar research was mainly an extension and consolidation of that carried out during the war years. In 1946, McPherson, Levy and Zibritosky [61] extended the work of Rafel and Sandlin [46] to compressed, Z-section stringer stiffened, flat panels subjected to normal pressure. Their test rig included an air cell and back-up structure with which they were able to apply positive and negative pressures to the surface of the panel. These pressures ranged from +16 psi to -8 psi. Their results demonstrated little of current significance beyond the fact that pressure positive outwards gave a stabilizing effect and pressure negative inwards, of course, tended to destabilize. The flat panel work of Rossman, Bartone and Dobrowski [30] was likewise extended by some 250 additional tests by



Schutte, Barab and McCracken [62]. In this work, the normal procedure of flat loaded ends and unloaded edges stabilized with stringers was followed. Only unanalyzed data was presented in this report.

A significant attempt to break away from this convention and introduce more definable conditions along the unloaded edges was made by Farrar [63]. He departed from both the stringer method introduced by Newell [12, 13], the guide system of Cox [28, 56, 52] and the V-groove method of Schuman and Back [11]. He designed special guide systems to give simply supported and clamped boundaries. His methods are clearly delineated in Figure 9.

The simply supported edge conditions were to be achieved by using hardened steel balls, one-fourth inch in diameter, held in V-grooves which were cut in steel blocks. To prevent the buckled plate from forcing apart the lines of balls along its edges, these blocks were bolted together by bolts perpendicular to the plane of the plate as well as being bolted to a steel section. It was intended that the balls would roll freely vertically in the V-grooves to permit unconstrained contraction of the plate. Little lateral contraction was expected in the plane of the plate so the reinforcement was designed to provide the maximum restraint normal to the plane of the plate. The balls were separated by rubber blocks to a five-sixteenth inch pitch. In this manner, if the loading head were to contact the uppermost ball, the load to compress the balls into the rubber blocks would be negligible.

In these tests, Farrar measured the panel mean strain with an

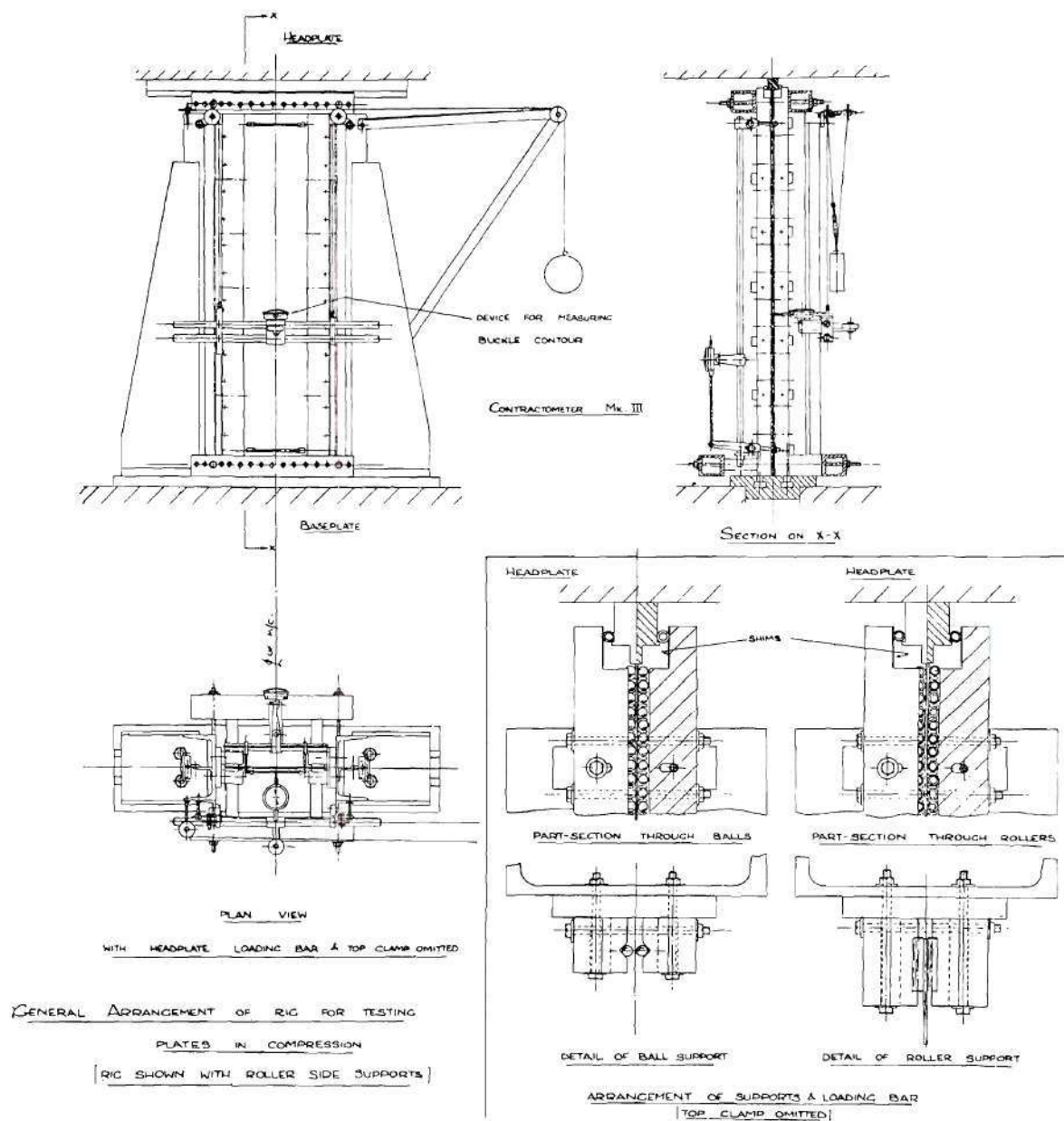


Figure 9. Test Fixture Used by Farrar (after Farrar).



averaging contractometer. Since the contractometer readings tended to become unreliable at large strains, dial indicators were used to measure the relative motion of the headplate and baseplate. Additionally, a dial indicator was mounted on a traversing frame so that the amplitude of the buckles could be determined from these measurements. He obtained the buckling strain in two ways. One method was to define the strain at which a sudden reduction in slope occurred in the load versus strain diagram as the buckling strain. The other method was to plot the square of the buckle mean amplitude versus strain as was done by Dunn [26]. The graphs were essentially linear and their intercept on the strain axis gave the buckling strain. The buckling strains so determined were converted to buckling stress coefficients which were compared with the existing theoretical values.

The comparison between theoretical and experimental buckling stress coefficients showed that simply supported (pinned) edge conditions had not been obtained. Farror felt the perturbation was due in part to the strips of plate which existed outside the ball supports. These strips were about 10 percent of the panel's width and acted both as elastic angular and lateral constraints at the edges. Two other effects were also presented:

- (a) The panel was of finite length and in general the half wavelength was not equal to the panel width, and
- (b) The ends of the panel which were in contact with the loading platens were constrained against lateral motion.

In order to simulate clamped edge conditions, hardened steel rollers one-fourth inch in diameter, were located in slots cut vertically in steel blocks. Rubber blocks were used to maintain these rollers at a five-sixteenth inch pitch. The steel blocks were bolted together perpendicular to the plane of the panel and bolted to the side supports as in the simply supported case.

Again, evaluation of the test results showed that the desired edge conditions had not been achieved. Farrar summarized the effects present to be:

(a) Incomplete clamping due to the deformability of the rollers, flexibility of the side blocks and initial lack of fit.

(b) Lateral constraint from the strips of metal clamped between the rollers, and possibly lateral frictional forces at the rollers.

(c) Lateral constraint at the ends of the panel in contact with the loading plattens.

(d) The finite length of the panel.

(e) When the ends are machined accurately plane, they tend to be clamped at the loading plattens.

In conclusion, Farrar stated:

"Tests with ball edge supports have not imitated pin edge conditions owing to the torsional stiffness of the plate material outside the supports; tests with roller edge supports have not initiated clamped edge conditions owing to the flexibility of the rollers."

Of course, the procedure of machining flat the loaded edges and stiffening the unloaded edges with stringers continued. It was

extensively utilized by Dow, Hickman and McCracken, [64, 65, 66, 67, 68]. Their investigations were directed toward strength comparisons between aluminum and magnesium sheets with aluminum stiffeners, the effect of rivet diameter and pitch on the strength of panels with Z-section stringers, and the strength of panels stiffened by Y-section and hat-section stringers. This work was noteworthy primarily as a result of the number of panels tested, in excess of 800.

In 1948, Schuette [69] announced the results of a test series initiated in 1943. In toto, 87 curved, unstiffened magnesium alloy plates with aspect ratios near unity, were tested. These specimens had their loaded edges machined flat and, for some curvatures, held by templates. The unloaded or lateral edges were "pinched" between flats or held by knife edges as shown in Figure 10. The emphasis in these tests was on the relationship between the buckling stress and the Secant Modulus. As is characteristic of curved panel tests, there was exhibited a high degree of scatter in the data on critical strain versus the thickness to radius ratio. A straight line, whose equation involved the Secant Modulus, was fitted through the data so that it was a lower bound to the region of greatest density of test points. The line was in the lower one-third of the scatter band and gave equivalent accuracy within and beyond the elastic limit.

In the studies that have been discussed, several researchers, particularly Cox [16, 28, 56] and Farrar [63], have devoted a considerable amount of time, thought and effort to the development of means to impose a desired set of boundary conditions upon the unloaded edges

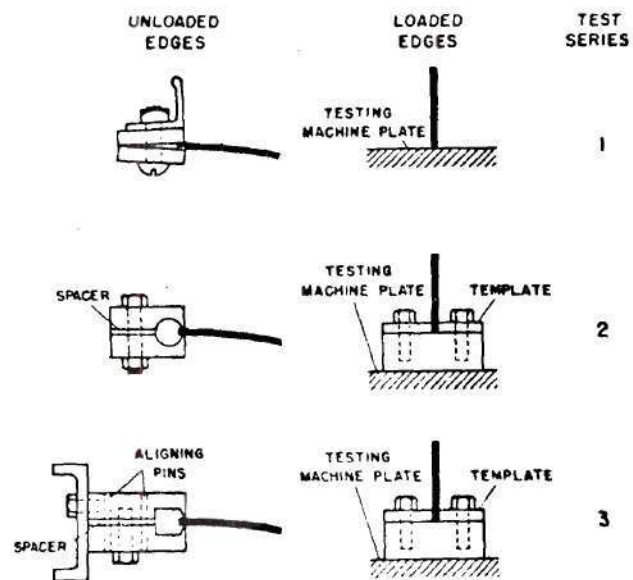


Figure 10. Methods of Edge Support  
Utilized by Schuette  
(after Schuette).



of their specimens. However, with regard to the loaded edges, the vast preponderance of specimens have either been machined flat, cast in a low-melting point alloy or securely clamped between angles.

A new device for simulating simple support on the loaded edges was introduced by the Forest Products Laboratory in 1947. The loaded edges of the panel were inserted into a series of slotted rods supported by roller bearings. Thus, the loaded edges of the panel were permitted to rotate as the axial compression was applied. This device, shown in Figure 11, appears to have a distinct advantage over the usually adopted V-grooves because it readily permits rotation. Its significance is that it facilitates simple support at the loaded edges even of thick panels, such as sandwich construction specimens. Consequently, it broadened appreciably the number of cases that could be investigated experimentally.

The studies at the F.P.L involved the testing in compression of flat sandwich panels of various sizes as well as combinations of two facing materials and two core materials. A total of 320 panels were tested, 80 in each of the following four combinations of boundary conditions:

- (a) All edges simply supported.
- (b) Loaded edges simply supported and lateral edges clamped.
- (c) Loaded edges clamped and lateral edges simply supported
- (d) All edges clamped

In the first series of tests [70], all edges simply supported, the device shown in Figure 11 was used along the loaded edges. The

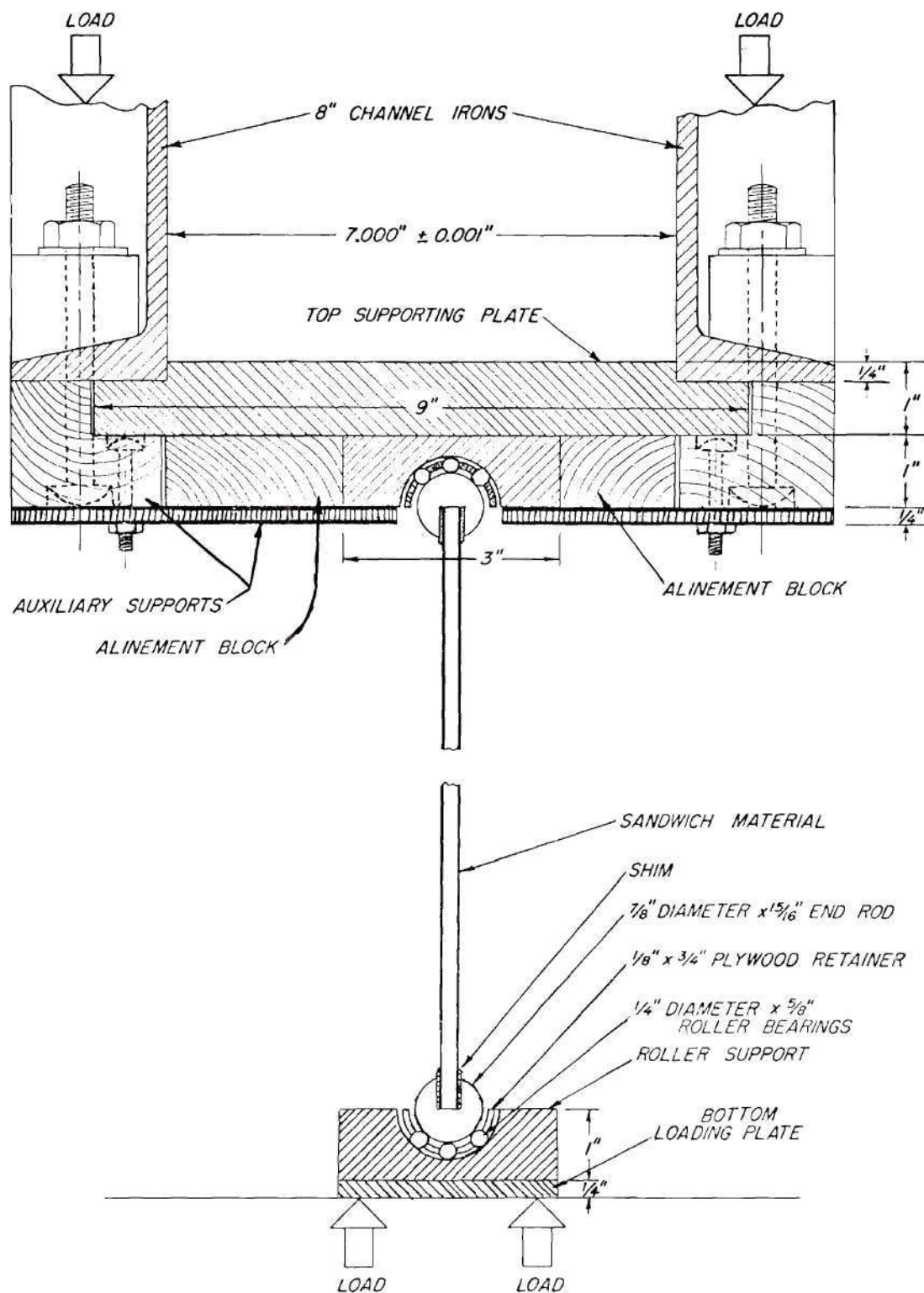
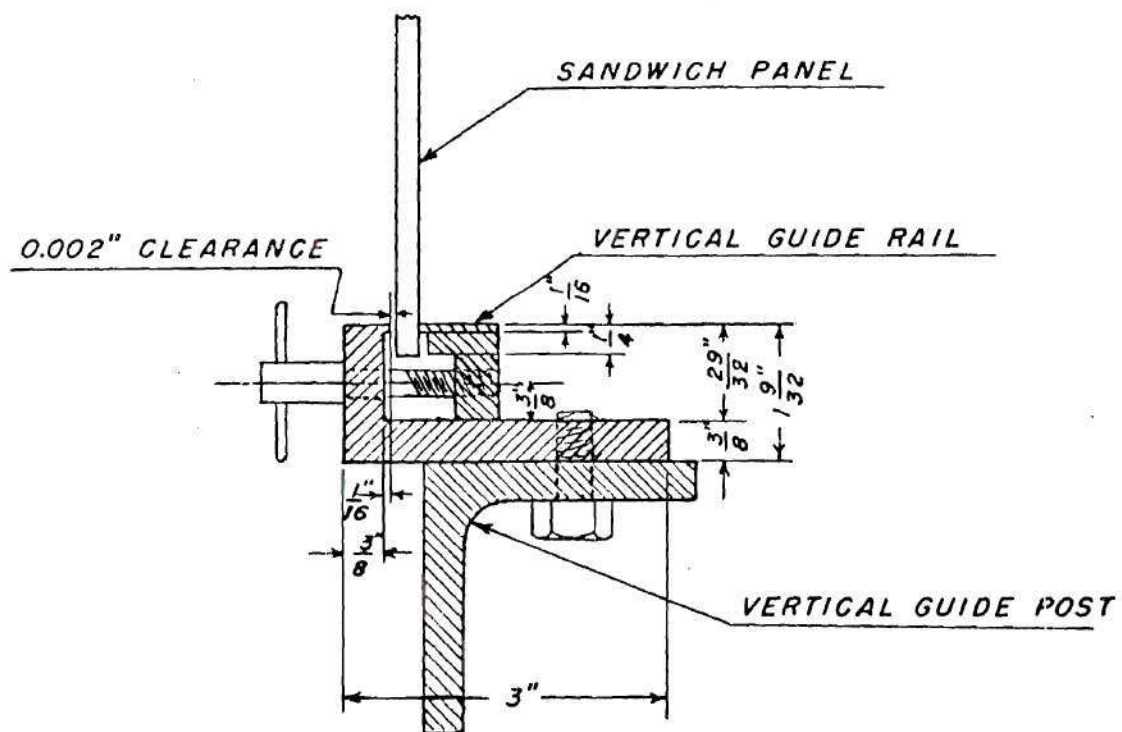


Figure 11. Fixture for Simply Supported Loaded Edges (after Boller).

unloaded or lateral edges were supported by vertical guide posts and rails as shown in Figure 12. The second series of tests [71], loaded edges simply supported and lateral edges clamped, retained the same device along the loaded edges as was used in the first series. The unloaded edges were clamped in the fixture shown in Figure 13. The third series [72] had the loaded edges clamped in the fixture shown in Figure 14 while the unloaded edges were simply-supported, as in the first series, by the device shown in Figure 12. The fourth and final series [73] employed the loaded edge fixture, Figure 14, of the third series and the unloaded edge fixture, Figure 13, of the second series. A series of five stiffened flat plywood panels [74] were also tested in axial compression with edges simply supported. The edge fixtures shown in Figure 11 and 12 were used in this case.

In 1948, Hoff, Boley and Coan [75] to avoid expensive grinding operations on the loaded edges of their test specimens refined the F.P.L. scheme. They accomplished this by inserting a flexible loading strip which would conform to the irregularities in the specimen's edges. The horizontal edges of the flat rectangular fiberglas panels were supported and loaded by a series of slotted rods which were each housed in their own individual nest of needle bearings. These needle bearings were mounted in steel blocks attached to the flexible steel strip. Then, the entire assembly was supported within and guided by a slotted base block. The compressive load was transmitted from the base block to the bearing assembly by means of screws mounted vertically in the plane of the test specimen. Using these screws, the strain distribution



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Figure 12. Lateral Edge Guides to Simulate Simple Support (after Boller).



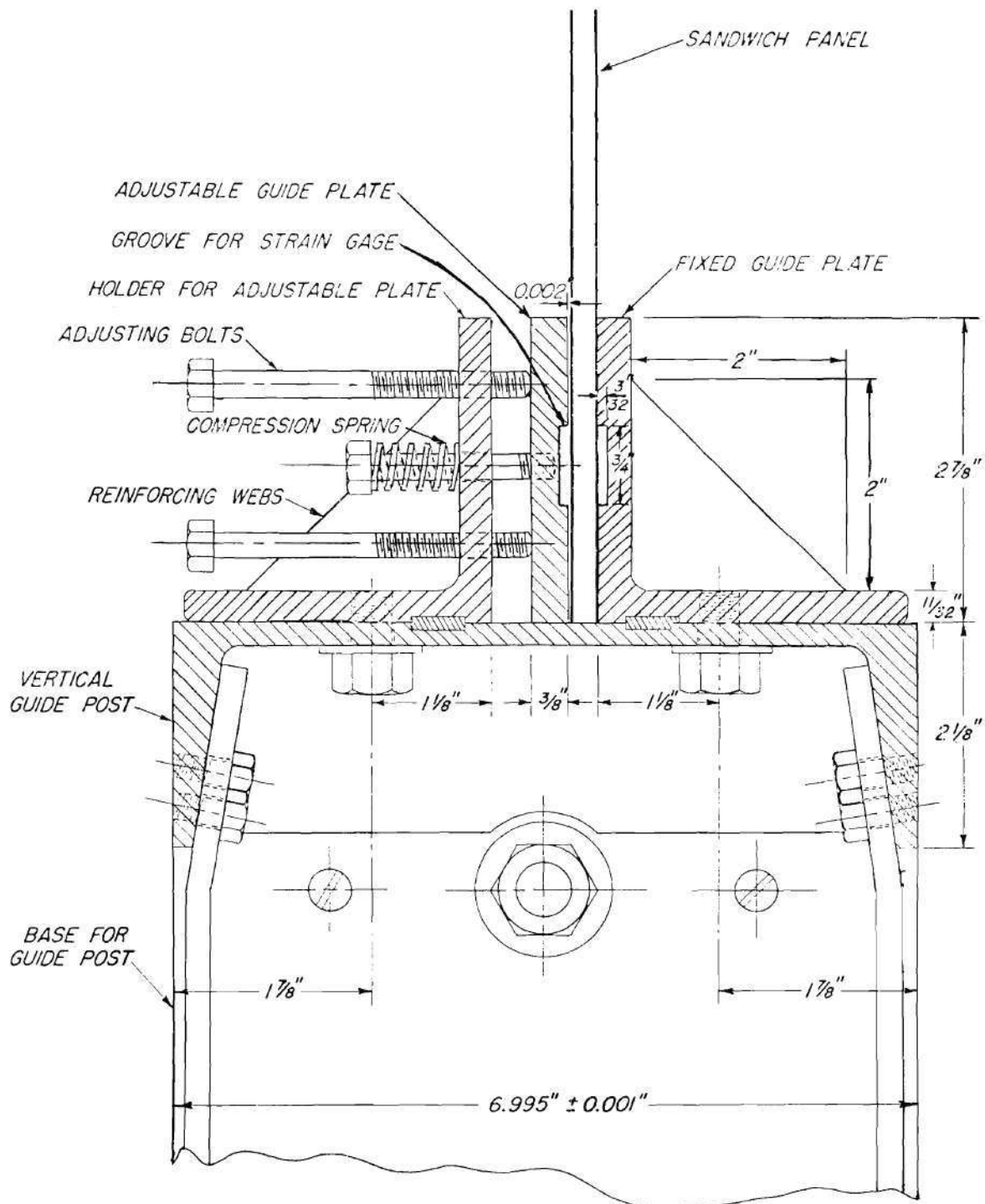


Figure 13. Details of the Fixtures That Provided Clamped Support (after Boller).

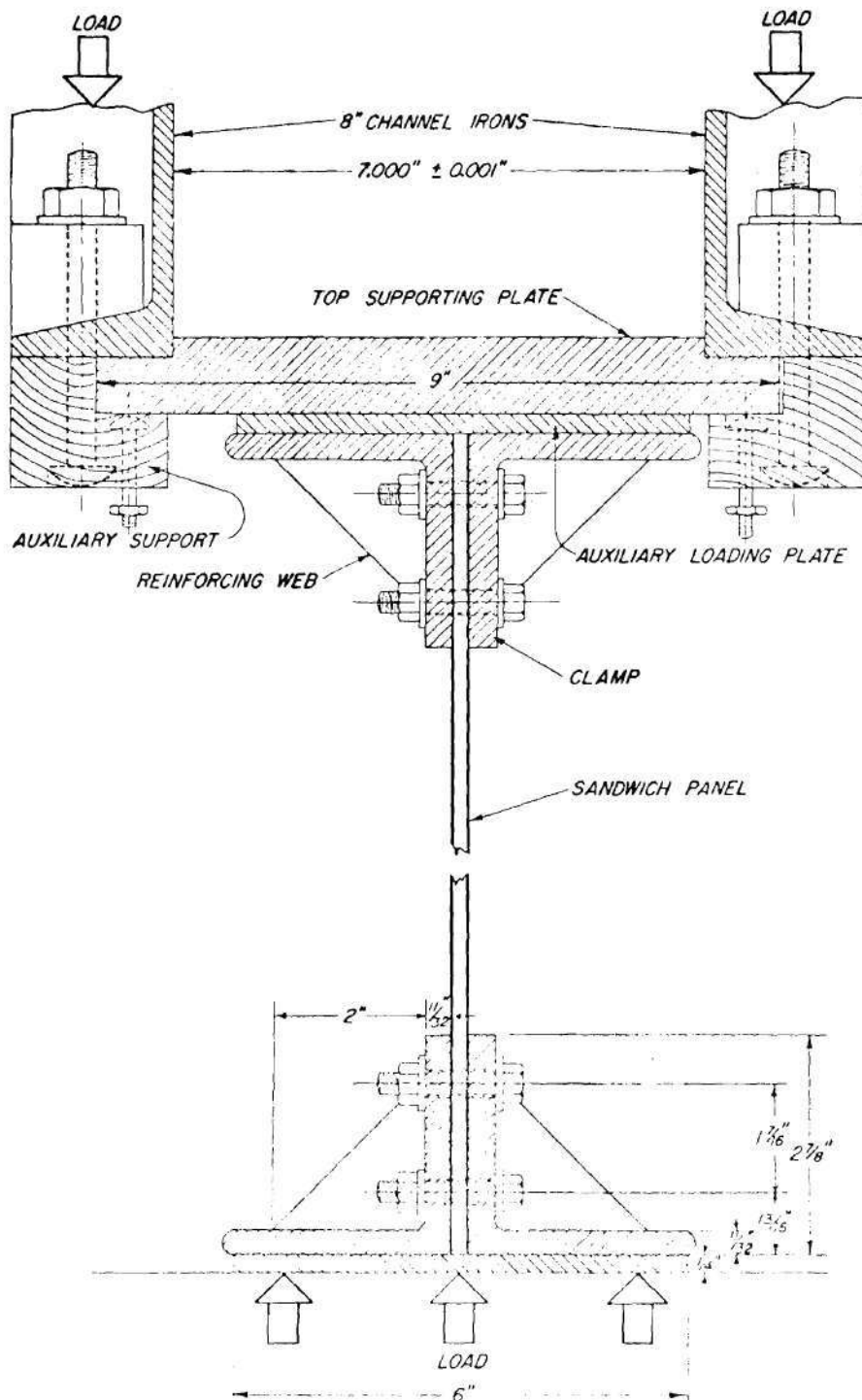


Figure 14. Sketch Showing the Clamping Fixtures at the Loaded Edges of the Panel (after Boller).

could be adjusted simply. With the specimen under axial load in the test machine, the strain gages on the specimen were read. Then the screws were tightened or loosened until the strain gages indicated a uniformity of strain distribution.

In this fixture, differential rotation of the loaded edges was permitted by the needle bearings and slotted rods. These needle bearing and slotted rod assemblies were closely spaced (one-eighth inch between adjacent assemblies) so that the specimen was supported for 91 percent of each loaded edge. There may well be a slight difficulty with the scheme. Uniformity of axial load distribution could be bought at the expense of edge restraining variation. The centers of rotation of the different portions along an edge are no longer colinear.

Two methods of support were employed along the unloaded edges; knife edges and slotted tubes. The slotted tubes were found to entail greater problems in assembly and alignment than the knife edges. Some of the difficulties in using the slotted tubes were that the relative position of the tubes had to be maintained by cross braces and then the tubes and braces supported until the specimen was under load. Also, the specimen had to be carefully plumbed. Finally, the contact pressure between the tubes and the specimen was uncertain when the slot was drawn up against the specimen by the bolts passing through each slotted tube. Thus, the use of the slotted tubes was discontinued. The knife edges were clamped to side supports of the rig. They provided a ready and uncomplicated means of mounting and aligning the

specimens in the rig. Load transfer from the specimen to the knife edges, although a possibility, appeared to be small and was neglected.

In 1949, the Forest Products Laboratory published a supplement [76] to their wartime reports on the buckling of flat plywood plates. These additional investigations were made on flat plywood plates whose grain direction was at  $45^\circ$  angle to the edge. In the tests the loaded edges were clamped and the lateral edges simply supported. The apparatus was quite simple. The loaded edges of the plates were clamped between pairs of steel bars by bolts passing through slots in the plates. The upper pair of clamping bars was attached to the head of the testing machine while the lower pair was supported centrally by a transverse one-half inch diameter steel roller resting on the test machine lower platten. This scheme provided longitudinal tilt freedom. The lateral edges were supported with three-sixteenths inch deep dove-tail grooves cut into one by one and one-half inch hardwood rails. These rails were made one-eighth inch shorter than the test specimens so that they would not carry any axial load. A curve for determining the buckling loads of plywood plates with this grain direction was then obtained by the energy method and fitted through the data points.

In order to determine the effect of curvature on the strain distribution around a circular hole in a panel Kroll and McPherson [77], in 1949, tested a set of 14 curved, unstiffened panels in axial compression. Identical specimens were tested; half with reinforcement, a doubler plate riveted around the hole, to evaluate the effectiveness



of the reinforcement. When doubler plates were placed on both sides of the panel, each doubler was the thickness of the sheet. When only one doubler plate was used, it was twice the sheet thickness. The outer radius of the doubler was such that it had a volume equal to twice that of the material removed by the hole. All the panels were 24 inches in width and 14 inches in length with a single central circular hole, three inches in diameter. The ends of the panels were ground flat and centered on ground steel blocks. Plaster of Paris was then cast between the steel blocks and the test machine heads to take up any lack of parallelism between the panel and the head. In this way, uniformity of axial load was sought. The unloaded edges of the panels were held in grooved bars. Electrical resistance strain gages were attached to the panels in back-to-back pairs. Dial indicators were used to measure normal deflections of the panels. The shortening of the hole diameter was also measured.

The tests indicated that, for panels without reinforcement, the mid-thickness strain distribution, in the sheet away from the hole, could be predicted by the plane stress theory for the corresponding flat plate. For panels with reinforcement, the plane stress theory values were lower than those observed in the sheet and higher than those observed, from the strain gage readings, in the reinforcement. This discrepancy was attributed to the reinforcement not being integral with the panel and thus not carrying its portion of the load. This was substantiated by the fact that the observed shortening of the hole diameter was essentially the same with or without the doubler.

During this period, new methods of construction were being explored in an effort to lower airframe construction costs. One method which held promise was the use of integrally extruded skin and stiffener panels. Dow and Hickman [78] evaluated the structural potential of such panels manufactured by riveting together three magnesium alloy sections of integrally extruded skin and T-section stringers. Their axial compression tests were conducted in the standard way, viz., loaded edges machined flat and stringer reinforced unloaded edges. They concluded that such construction should be further investigated since the load carrying ability of structures so fabricated was comparable to conventional ones.

They also published the results of three other investigations on panels [79, 80, 81]. The first series of tests, 48 panels with curved-web Y-section stringers, was conducted for comparison of their structural efficiency with that of straight-web Y-section stringer panels tested previously. Panels with Z-section stringers were the subject of the other two test series. One set, 300 panels, was tested to study variations in the ratios of stiffener thickness to skin thickness and stiffener spacing to skin thickness. The other set, 364 panels, was tested to ascertain the effects of variations in thickness and spacing when the stringers were small and widely spaced. A synopsis of the test techniques used and the influences and variations examined in these various tests at the NACA Langley Structures Laboratory was given during this time by Dow [82]. Another paper which catalogues these various tests by stringer section is that of Gabrielli,

Massa and Sacchi, [83]; however, it is in Italian. Of course, due to changes in design philosophy and manufacturing techniques, these specimens and consequently the results are not as pertinent today as they were in the late 1940's.

#### Research from 1950 through 1959

In 1950 and 1951, panel buckling studies continued at the F.P.L. In one such study [84], the effect of a central circumferential rib on the buckling of curved plywood panels was investigated. One hundred and five specimens of various lengths, widths and radii were used. These specimens were prepared in a similar fashion to those of the previous year, viz., loaded edges finished smooth and unloaded edges supported in wooden rails. A typical panel is shown in Figure 15. Rotational freedom was obtained by using a spherical bearing rather than a roller. Flat sandwich panels under axial compression were also tested [85]. The vertical guide rails, shown in Figure 12, were used on the unloaded edges while the other edges were loaded through steel bars. Combinations of three facing thicknesses and two honeycomb cores were utilized in the fabrication of these specimens.

Metal sandwich construction panels were being investigated by the N.A.C.A. in 1953. In this instance, Kroll, Mordfin and Garland [86] were concerned with the combined loading condition of axial compression and lateral pressure. Although sandwich panels with balsa wood cores had been tested before this time, these tests were among the first made on metal specimens containing a metal hexagon honeycomb core.

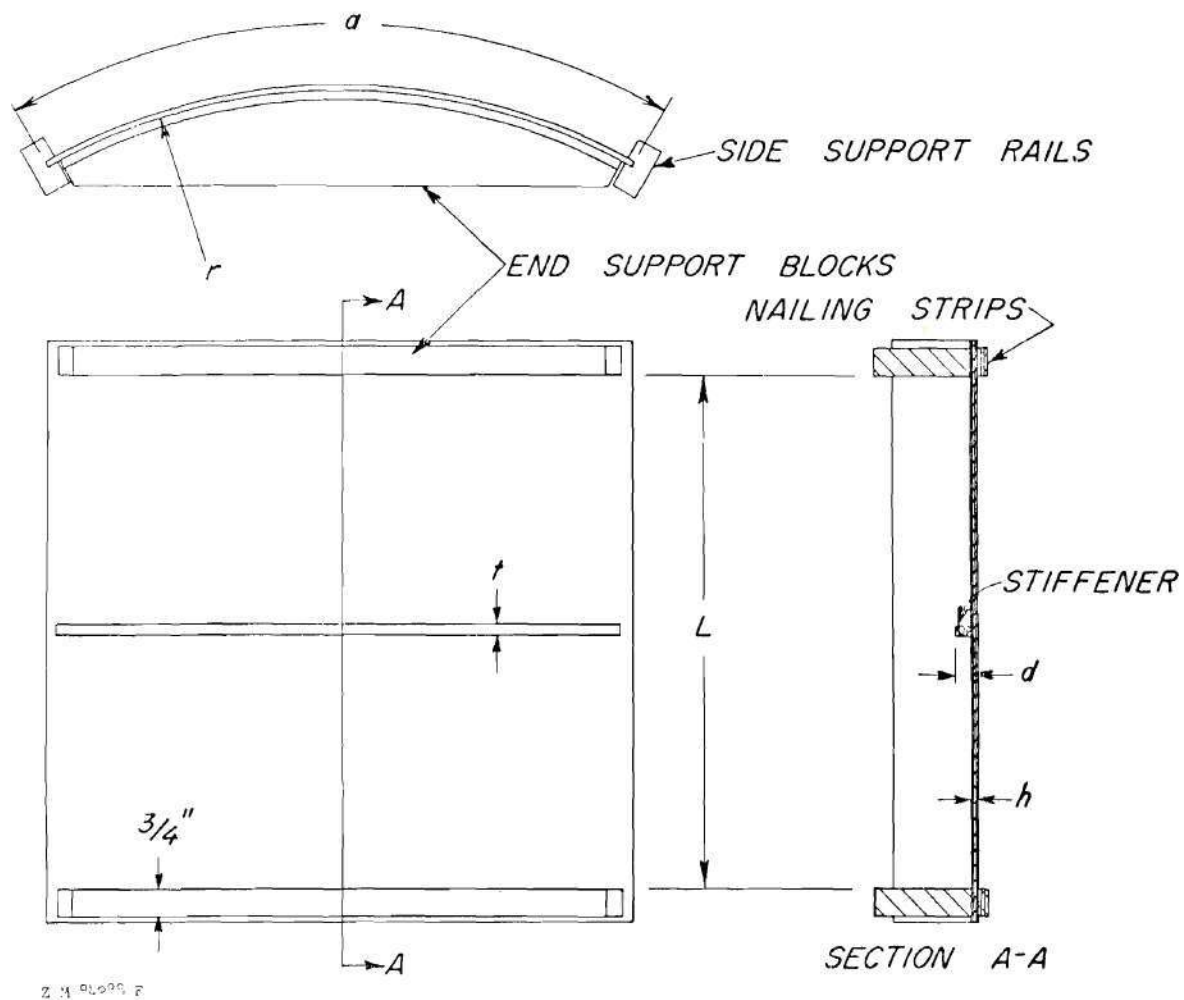


Figure 15. Curved Panel Specimen Tester at Forest Products Laboratory (after Heebink and Norris).



After several dissappointments, a successful apparatus was developed for the tests. Simply supported loaded edges were simulated by using knife edge fixtures. These fixtures consisted of bars with slots into which the panel's loaded edges fitted. On the bottom of each bar directly beneath the panel, there was a triangular knife edge. The knife edges rested in V-grooves machined into bearing blocks. To accommodate the rotation of the knife edge fixtures a larger groove was machined beside the V-groove. Since only one side of the specimen would receive lateral pressure, the direction of panel deflection and hence knife edge fixture rotation was known a priori. The loaded edges of the specimens were machined flat and parallel before insertion into the knife edge fixtures. The unloaded edges of the specimens were free. Lateral pressure was supplied by an air bag between the specimen and a back-up structure which was designed to carry no axial load. It was concluded from the 12 specimens tested that failing loads of sandwich panels could be predicted satisfactorily with existing theory.

As previously noted with regard to extruded panels [78], new methods of construction were being sought so that manufacturing costs could be lowered. By the early 1950's, metal-to-metal bonding by epoxy adhesives had progressed to the stage of possible aviation application. Such bonding held promise of relative ease and economy in the fabrication of stiffened panels. However, the relative merits of bonding and of riveting in such panels had to be ascertained by testing before decisions could be made as to utilization.

In a thesis at The College of Aeronautics, Cranfield in 1953,

Labram [87] presented the results of axial compression tests on 12 pairs of flat panels. The objective was to compare bonding and riveting as stringer attachment methods. The panels, stiffened by three hat-section stringers, were combinations of three sheet thicknesses and four spacings of stringers. The loaded edges were cast in a low-melting point alloy and then machined flat and parallel. The unloaded edges were held between wooden strips as was done by Gerard and Dickens [18].

Tests of a similar nature were conducted in the United States during the next year. Mordfin and Wilke [88] utilized three techniques for panel construction; namely, Z-section stringers riveted to the sheet, I-section stringers bonded to the sheet by Araldite adhesive and I-section stringers bonded to the sheet by Metlbond adhesive. Each panel was stiffened by five stringers, had its loaded edges ground flat and parallel and its unloaded edges stiffened by stringers. In six specimens, load diffusion was studied by having the axial compression applied through the central stringer which, in these specimens, protruded beyond the sheet. The ability to spread a concentrated load was found to be nearly the same for the riveted and bonded specimens.

The results of Labram were similar to those obtained by Mordfin and Wilke. The two series of tests showed that the static strength properties of both riveted and bonded panels were comparable. The difference in failure modes, however, might make the bonded panels less desirable in certain applications. The bonded panels failed with a wide spread separation of the stringers from the sheet. On the other

hand, the riveted panels failed by local buckling of sheet and stringers between rivets.

In an attempt to discover if there was a contribution from changes in material properties on the compressive strength of flat stiffened panels, an investigation was carried out by Dow, Hickman and Rosen [89]. Panels were constructed from nine different materials to obtain variations in yield stress and Young's modulus. The effect of these variations on the average stress at maximum load, the local buckling stress, and the load-shortening characteristics of the panels was determined. All of the panels were compressed flat-ended and without side support. However, the hydraulic test machine was modified so that, as its crosshead applied load to the specimen, load was also applied to a hydraulic jack. By this modification, as the specimen reached maximum load, the continued motion of the crosshead was opposed and controlled by the auxillary jack. In this way, even beyond the maximum load, the load-distortion characteristics of the specimen could be observed. This modification was necessary since a hydraulic test machine is a "dead weight" loading device. Thus, collapse of the specimen under such loading must be prevented if the post-buckling behavior is to be fully investigated.

During this and several preceding years, the traditional methods of construction were constantly being augmented by new techniques. Extruded integrally stiffened panels and panels with bonded stringers have already been cited [78, 87, 88]. Another new technique was that of "waffle" stiffening. In this method, the sheet was stiffened by



integrally attached crisscrossing stiffeners. These stiffener grids could be manufactured either by casting, by using a milling machine or by "chem-milling", a process by which even intricate shapes of metal can be removed by acid. Nine configurations of integral waffle-like stiffening were tested by Dow, Levin, and Troutman [90] in 1954. Stiffened plates, 45 inches in length and 10 inches in width, were fabricated as sand castings then machined to the proper thickness and riveted together with corner angles to form square caissons. The tests of these nine caissons were, to the best knowledge of the author, the first experiments made in the United States on the buckling of panels with skew-orthogonal and non-orthogonal integral stiffening. It should be mentioned that although such integral stiffening was new, non-orthogonal stiffening was not. As a specific example, the Wellington medium bomber, developed by Great Britain prior to World War II, had a fuselage of "geodetic" construction, viz., a non-orthogonal lattice of metal covered with fabric [91].

Other work on panel buckling conducted at the Langley Structures laboratory of the NACA during this period was published by Holt [92] and also by Stein [93]. Holt's work was a comparison of the compressive strength of flat, stiffened panels manufactured from three different aluminum alloys. The panels were stiffened with U-section stringers and tested flat ended without side support. In an attempt to provide simply supported unloaded edges, Stein utilized a multiple bay system. An unstiffened flat plate, 52 inches wide and 25 inches long was divided into 11 bays, each 4.7 inches wide and 25 inches long by



by multiple pairs of knife edges. These knife edges were lubricated at their points of contact and were able to rotate in order to facilitate in-plane movement of the plate. The loaded edges of the plate were ground flat and parallel. Only the middle bay of the 11 bays had instrumentation, 8 pairs of back-to-back wire-resistance strain gages were placed along the center of the plate. This work is significant due to the excellent agreement between the stability load attained by experiment and that predicted by the theory for simple support conditions.

In view of the large amount of panel test work which was done prior to 1956, it is surprising that the question of scaling did not arise. Either because facilities were lacking or because it was assumed that scaling effects were negligible, no investigations were conducted into the applicability of small scale tests to full scale vehicles until 1956. That year, Doman and Schwartz [94] sought to determine whether or not there were significant effects due to the size of the specimen. In their study, they utilized the five million pound capacity test machine at the U. S. Navy Aviation Structure Laboratory in Philadelphia. Three Z-section stringer stiffened panels were constructed in each of four configurations. These panels varied from 172 to 107 inches in length and were from 62 to 40 inches in width. An equal number of geometrically similar, one-fourth scale, panels were also constructed. All panels were tested flat ended and without side support. Reassuringly, it was found that there was no significant effect due to size.

Research from 1960 through 1969

The first of a series of reports on stiffened steel plate experiments by the Fritz Engineering Laboratory of Lehigh University was published in 1960. The research, under the sponsorship of the U. S. Navy, spanned the years from 1960 to 1964 and was carried out on specimens representative of ship construction. The initial series of tests [95] was made upon five panels of A-7 steel, one-fourth inch thick, 60 inches long by 51 inches wide, and had four longitudinal, steel, T-section stringers welded to the plate. The fifth specimen had closed section stringers formed by welding steel strips between the plate and the tops of the T's. The panels were tested under combined axial compression and lateral pressure with the pressure applied by an air cell held against the panel face by an auxiliary structure. The loaded edges of the specimens were attached to rectangular steel end blocks on which were cylindrical bearing bars. These cylindrical bearing bars were of the same width as the panels and bore on platens fastened to the test machine. They were used to simulate a simple support condition along the loaded edges. Since there was a T-stringer at each lateral edge, no side support was used. The second series of tests, [96], was conducted in 1962. Five specimens of the same design as those used in the first series were subjected to axial compression alone. One panel, stiffened by six stringers, was tested under lateral pressure and axial compression. The test fixture was the same one that had been employed in the first series, i. e., simply supported along the loaded edges and unsupported along the lateral edges.

The third series of tests [97], reported in 1964, involved four panels of the same design as the previous ones. These were loaded with combined axial compression and lateral pressure. A fixed boundary condition was simulated along the loaded edges by having the specimen end plates bolted to blocks which were ground flat and parallel to the platens of the test machine. As in all previous tests, the lateral edges were supported by a stringer.

Soderquist [98] of the U.T.I.A. published the results of experiments on the stability and post buckling behavior of axial compressed, curved and stiffened panels. Each of the 19 panels, 14 curved and 5 flat, were stiffened by two rectangular cross-section stringers. The loaded edges of the panels were cast in Wood's metal and then machined flat and parallel. Simple support was simulated along the unloaded edges by holding these edges between two angles which were separated by a shim equal to the panel's thickness. He detected the onset of buckling by using the photogrammetric method of Jackson and Hall [51]. He concluded that the ultimate load capacity of curved stiffened panels was strongly dependent upon the curvature parameter,  $Z_b$ , introduced by Batdorf, [99, 100] and upon  $b/t$  (the panel's width-to-thickness ratio). The effective widths were also strongly dependent on  $Z_b$ . They followed the theoretical predictions of Sechler and Dunn [101] but were consistently conservative. He also stated that the instability coefficients, derived from his test data showed no consistent trend with  $R/t$ , (the panel's radius-to-thickness ratio).

Soderquist's study was continued the next year by Tennyson

[102] of the U.T.I.A. In his two test series, he used the same boundary conditions as Soderquist. In his first series of five curved and three flat panels, the two open-section stringers on each panel were attached by bonding. In the second series of five curved and five flat panels, the stringers were fabricated, in various cross-section shapes, from photoelastic plastic and were attached by epoxy. From these experiments, Tennyson concluded that the theory of Seide and Stein [103] was accurate for flat panels. He confirmed, too, the statement of Soderquist that the effective widths obtained from experimental data followed the trend predicted by Sechler and Dunn [101] and Wenzek [20], although the predicted values were conservative.

The early 1960's were marked by interest in beryllium as a structural material, because of its high specific strength. In 1961, Crawford and Burns [104] tested five flat, unstiffened beryllium plates in axial compression. Three of these were tested at room temperature (70°F) and the other two at an elevated temperature (680°F). The test apparatus was similar to that of Stein [87] but only four pairs of vertical knife edges were used across the plate's width. Bars of either lead (70° case) or annealed copper (680° case) were placed along the loaded edges of the plates to achieve uniformity of load distribution across the width of the panel.

Yamaki [105] in 1961, conducted experiments on the post-buckling behavior of square, aluminum, unstiffened plates under uniaxial compression. Four boundary conditions were examined.

- (1) All edges simply supported,



- (2) Loaded edges simply supported and lateral edges clamped,
- (3) Loaded edges clamped and lateral edges simply supported,
- (4) All edges clamped.

These boundary conditions were simulated by the use of knife edges bolted to the test fixture; one pair of knife edges holding the specimen's edge for the simply support case and two pairs of knife edges, side by side, holding an edge for the clamped case. From examination of the readings taken from the back-to-back strain gages on the specimen, Yamaki found that there was appreciable bending strain produced when a pair of knife edge was used to achieve simple support along the loaded edges. A pair of knife edges was satisfactory, however, when used to achieve simple support along unloaded edges. The arrangement of two pairs of knife edges for a clamped condition was found to be satisfactory along all edges. The load was measured with a dynamometer, a rectangular beam with strain gages attached to it, which was inserted in the load path. The magnitude of the applied load was determined from the bending strains in the beam. The out-of-plane deflections of the specimens were measured with two equal stiffness dial gages set opposite to one another. In this way, the equal spring forces cancelled and did not influence the behavior of the specimen. Yamaki's test fixture is shown in Figure 16.

During the early 1960's considerable attention was paid to the problems of high speed flight particularly those which occurred due to thermal effects. Mention has already been made of the stability tests carried out on beryllium flat plates at elevated temperature.

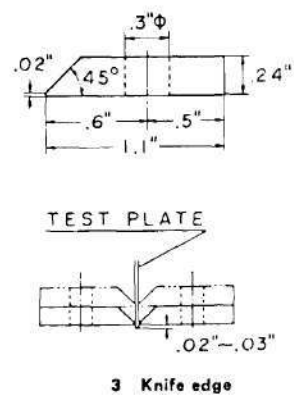
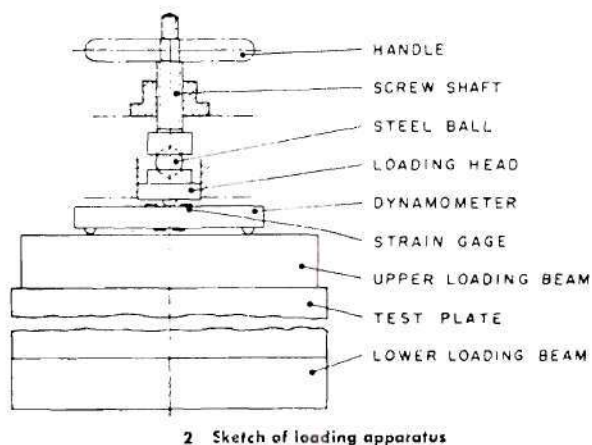
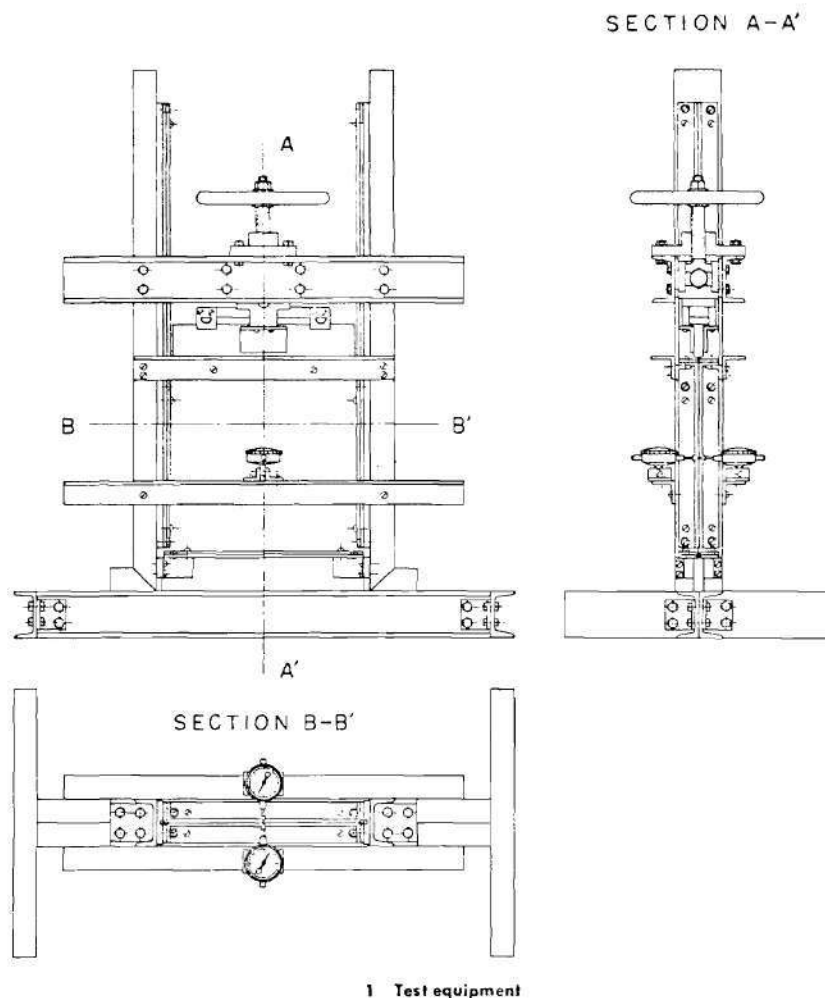


Figure 16. Test Equipment Used by Yamaki  
(after Yamaki).

In such an environment, sandwich construction also has some attractions. Thus, the effects of a thermal gradient between the facings of metal honeycomb panels became important. To investigate the issue, Chang and Timmons [106] selected aluminum and stainless-steel honeycomb sandwich panels due to their potential usage in aerospace structures. Their test fixture utilized basically the same method of achieving simply-supported along the loaded edges as had been used by Boller [70] at the F.P.L. The only modification was that ball bearings were used instead of roller bearings. Along the unloaded edges, the panel was fitted into a series of slotted disks. However, now the ball bearings were eliminated and the slotted disks rotated in tubular side channels. It found that the wave deflection pattern assumed in their theoretical development correlated well with the actual deflection pattern of the specimens.

A series of tests to ascertain the effect of ring spacing on the strength of curved panels in axial compression was conducted at North American Aviation and reported by Schleicher [107] in 1962. A number of steel panels were manufactured with a 28 inch radius of curvature. These panels had heavy main rings at a constant spacing and light intermediate rings whose spacing was varied. The unloaded edges of the panels were lightly clamped between angles but scant information was given as to the conditions at the loaded edges. It appears that the loaded edges were machined flat.

The influence of surface dents was also studied. In contrast to the work of Welter [59] where the surface dents were applied by the

impressing of a steel ball, the surface dents were applied by a hammer. The initial buckling load was reduced by approximately 25 percent but the post-buckling load by only 4 percent over those of an undamaged specimen. Welter, it will be remembered, obtained virtually no difference in the buckling load between dented and undented specimens.

Schleicher drew the following conclusions from these tests.

- (a) The initial strength of a curved panel depends on the initial surface conditions.
- (b) The post-buckling strength, except for the damaged panels, is less than the initial buckling strength.
- (c) Light, intermediate rings add greatly to the buckling strength of curved panels.
- (d) After initial buckling, the strength of the panels is consistent and reproducible.

It would be well to examine this last conclusion the light of the study of shell buckling conducted by Horton and associates [53]. In their study on unstiffened, electroformed nickel cylinders with internal mandrels, they found that the repeated buckling behavior of the same specimen formed a series of plateaus. This behavior, shown in Figure 17, is attributed to the damage sustained by the shell during its buckling, i. e., the amount of inelastic deformation. The internal mandrel, by limiting the inward deflection of the shell, was able to prevent severe inelastic action. By analogy with these shell tests, it would appear that the curved panels reported by Schleicher were exhibiting the shell behavior that appreciably curved panels should



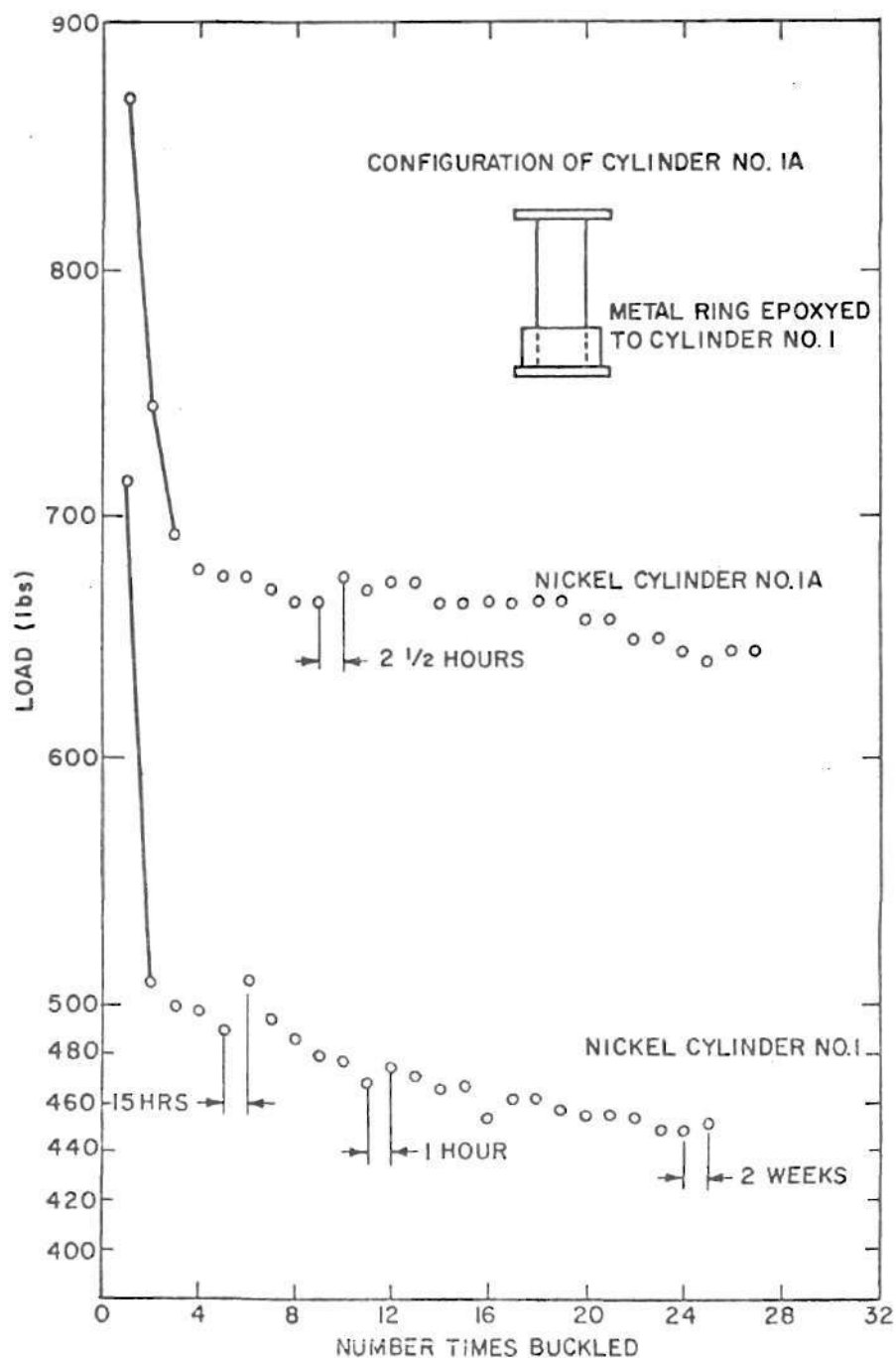


Figure 17. Repeated Buckling Behavior of a Cylindrical Shell (after Horton et al.).

show. The buckling loads in the shell tests, as Figure 17 shows, approach a repeatable value. This is a result of the plastic effects in the folds of the buckles. It can also be seen, in Figure 17, that there is a time effect such that the buckling load is slightly higher after the specimen is allowed to "rest".

The following year, interest was still unabated in the thermal problems of high speed flight which were plaguing the designer. Several possible solutions had been proposed and used but each had distinct merits and deficiencies. Titanium is capable of undergoing higher temperatures than aluminum but it is more expensive and quite difficult to fabricate. Ablating materials are the most successful form of protection for re-entering bodies but they must be replaced following each flight. A third solution is to install a cooling system, perhaps in conjunction with other forms of heat protection such as ablating materials and/or titanium. This appears to be particularly attractive for re-usable vehicles, i. e., space shuttle systems. One important question is whether or not buckling of a panel with attached cooling coils constricts the coils and thereby hampers system performance. Hence, in 1963, Dow and Whitley [108] tested six flat panels, stiffened with Z-section stringers, which had integrally formed or attached cooling circuits. For comparison two additional panels were manufactured without the cooling circuits. The panels were tested at room temperature with water flowing in the cooling circuits at the design flow rate and inlet pressure. The variations in these parameters due to the axial load were monitored. The loaded edges of the panels

were ground flat and parallel. Lateral restraint was imparted to the panels by bolting steel beams between the test machine and the two intermediate ribs on the panels.

Interest in beryllium as a structural material was again manifested in 1964. Rebholz [109] published the results of axial compression tests on 23 curved beryllium panels. A substantial portion of the research effort was used to devise suitable fabrication procedures for beryllium. Hence, three distinct methods were used in forming the specimens; heat-lamp creep forming, roll-forming, and closed-die creep-forming. Brazing procedures for beryllium were also tried for the attachment of two beryllium hat-section stringers to each of three beryllium curved sheets. The loaded edges of the panels were machined flat and parallel and then clamped between templates attached to the test machine plattens. The unloaded edges were held in side clamps of the same type as used in Schuette's [64] second test series and shown in Figure 10.

During the same year, Schlack [110] tested flat square plates with central circular holes of various sizes. To obtain all simply supported edges, he followed the lead of Boller [70], Hoff, Boley and Coan [75], and Chang and Timmons [106]. He obtained good agreement between the experimental and theoretical buckling loads for the pierced plates.

In 1967 Walker [111] published the results of a significant experimental and theoretical study of plate buckling. This was the first investigation to include experiments with a linearly varying distribution

of axial compression. The fixture used to impose this loading is shown schematically in Figure 18. In it, axial compression was applied by manually operated loading screws located on each side of the specimen. The linearly-varying distribution of compression was accomplished by differential rotation of the screws. The loaded edges of the specimens were held by slotted rollers nested in needle bearings. As stated previously in the discussion of the work at F.P.L. [70] as well as that of Hoff, Boley and Coan [75], this is a most satisfactory method of achieving simple support. In regard to the lateral edges, Walker stated that "The general theoretical condition of rotational restraint at the unloaded edges is extremely difficult to produce experimentally. For this reason only the limiting cases of the condition were studied, i. e., 'simple-support' and 'built-in'." The boundary restraint devices are shown in Figure 19.

Walker also encountered difficulties in the numerical portion of his study. The lengthy calculations entailed in the Galerkin process precluded consideration of more than a few cases. Walker's test fixture was later used by Brown and Harvey [112] in 1969 for plates subjected to lateral pressure and axial compression. The lateral pressure was applied by an air bag situated beneath the specimen. Both Walker, and Brown and Harvey utilized the photogrammetric method of Jackson and Hall [55] to obtain the out-of-plane deflections of the specimens. The buckling loads were determined by Southwell Plots in each of these studies.

A variant of the "comb" array developed by Cox [29, 57] was



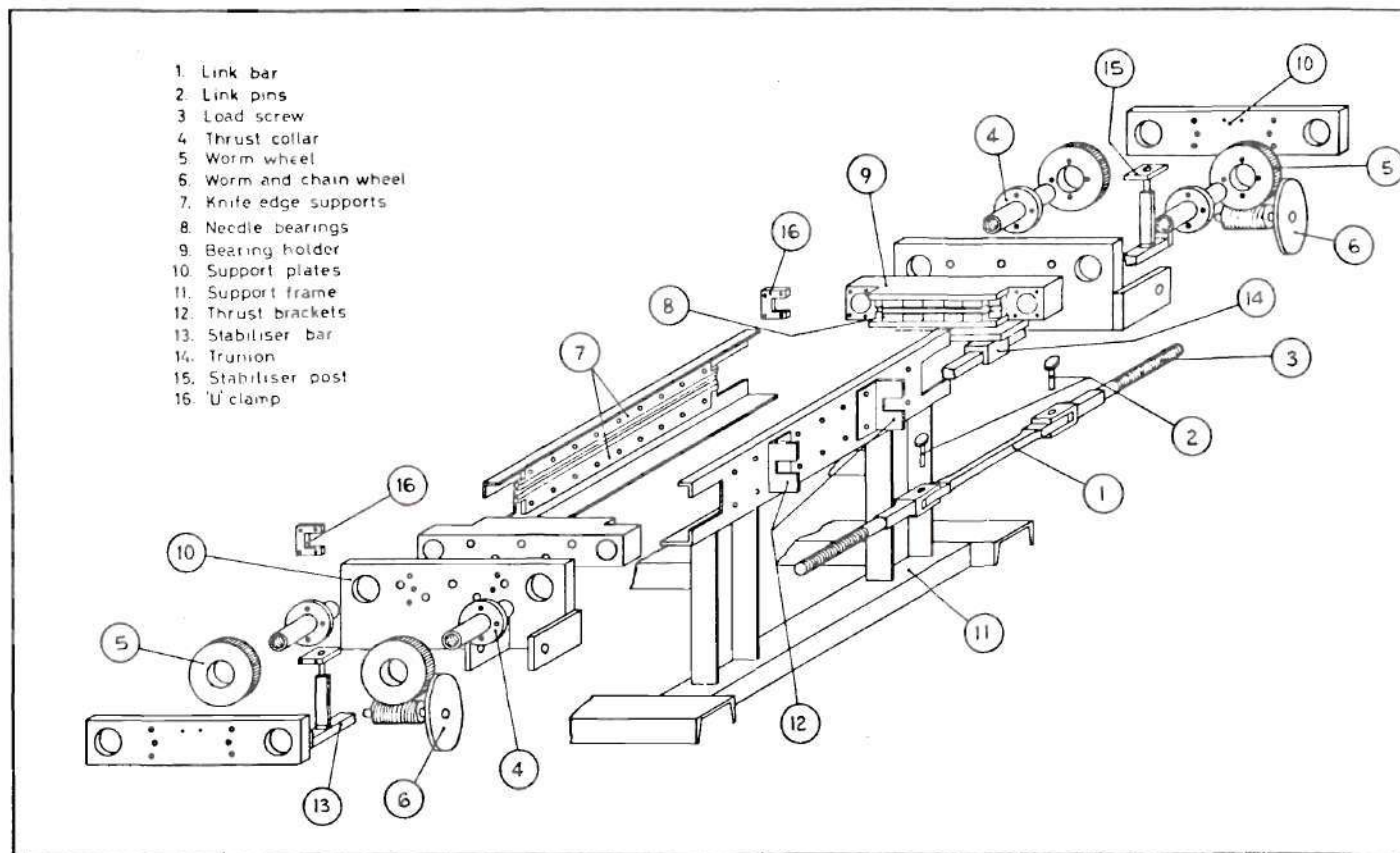
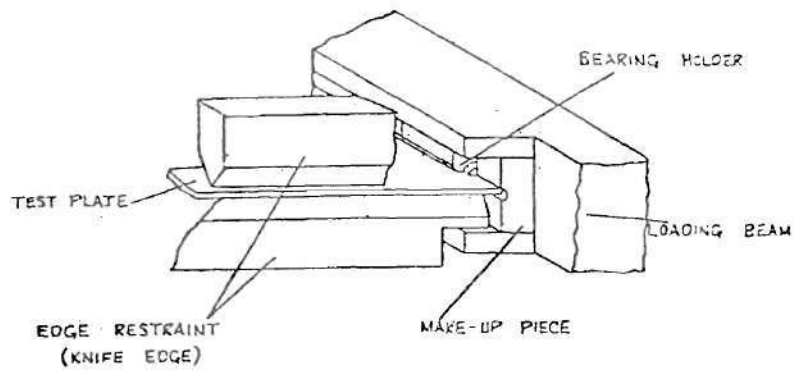
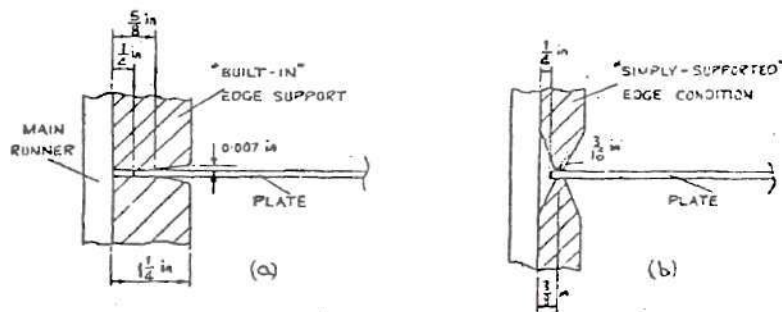


Figure 18. Components of the Plate Testing Rig  
 (after Walker).



Loaded-edge at the extremity of the plate.



Details of unloaded-edge supports.

Figure 19. Boundary Restraint Utilized by Walker (after Walker).

employed by Hoff, Levi and Honikman [113] in their study of creep buckling. In this test apparatus, the individual strips in the "comb" were cantilevered out from the fixture. Instead of the slot in each strip there was a V-notch which intersected, at its apex, a drilled hole. In this manner, the panel specimens were "simply supported" between the two spurs of metal formed by this intersection rather than "clamped" along the depth of a slot as Cox had done. Each unloaded edge of the specimen was supported by 180 such "fingers" of 0.015 inch thick inconel separated by 0.064 inch spacers. The loaded edges of the specimens were supported in V-grooves.

During the same time period, Sharman and Humpherson [114] were also conducting experiments on simply-supported plates under lateral load and axial compression. Their specimens were made from Perspex (a plastic similar to plexiglas) and the Moiré technique was used to determine the initial irregularities and the deformations due to load. The simple support condition was achieved by using slotted rods which were attached to the specimen edges and which rotated in machined grooves.

Another prevalent method of obtaining simple-support; namely, knife edges, was used by Ashton and Love [115] in 1969. They tested 20 laminated, boron-epoxy plates with various fiber orientations and, for comparison, four isotropic plates. The loaded edges were always clamped between steel blocks while the lateral edges were either simply supported or clamped. The buckling load was determined by the use of Southwell plots. They obtained good correlation between the

test results and the theoretical predictions.

#### Recent Research 1970 to Present

A rather interesting and novel test fixture was developed by Kicher and Mandell [116] to study the buckling of composite plates under uniaxial compression. Two sets of boundary conditions were treated experimentally:

- (a) loaded edges simply supported and lateral edges free, and
- (b) all edges simply supported.

In this fixture, the loaded edges of the specimens were inserted into slots in a series of wedge shaped bearing points. Each of these bearing wedges fitted into a V-grooved, spring supported piston. These piston springs were matched to the axial stiffness of the specimen. Thus, as the load was applied, the springs displaced relative to one another thereby producing a reasonably uniform load distribution. The unusual feature of the test apparatus was that it produced uniform load rather than a uniform strain. Along the lateral edges, the simply supported boundary condition was simulated by confining the edges between hardened steel rollers. Kicher and Mandell, like so many of their predecessors, used Southwell Plots to determine the buckling loads. For these plots, the deflections were taken at the point of maximum lateral deflection. As had been noted by Gough and Cox [117] over thirty years before, the point of maximum lateral deflection shifted during a test and, thus, necessitated moving the transducer to maintain contact with it. The agreement between



experiment and theory was good.

Quite recently, a test program was conducted at North American Rockwell [118] to evaluate the stability requirements for the ribs on the flat compression panels projected for the Space Shuttle. The loaded edges of the specimens were cast in epoxy and then machined flat and parallel. They were then rested on loading plates which had a cylindrical bearing block mating into a cylindrical seat along the full width of the panel. In this way, simple support was simulated at the loaded edges. Along the lateral edges, simple support was simulated by slit steel tubes squeezed onto them.

Tenerelli and Holmes [119] have published the results of the most recent panel tests of which the author is aware. They adopted several unique procedures. The most important was the minimization of the moments at the center of the specimen. They accomplished this by loading the specimen via an array of small hydraulic jacks and a tilting base plate. A hydraulic system was also used to provide appropriate reaction. To prevent edge failures the lateral edges were stiffened with angles. The data obtained was interpreted using the Southwell method and the results were in excellent accord with the theoretical predictions.

The most important experimental studies are summarized in Table 1. It is apparent that when the best techniques for testing and data interpretation are used, the agreement between theory and experiment is excellent. In view of this, the studies described in the subsequent chapters are entirely of a theoretical character.

Table 1. Major Steps in Experiments on Plate Stability

Date	Researcher	Specimen	Boundary Restraint	Method of Buckling Determination	Significant Features	Agreement With Theory
1847	Fairbairn & Hodgkinson	Caisson	Lateral Edges Simply Supported. Loaded Edges Flat.	Visual Inspection	First Recognition of Plate Instability	Theory Nonexistent
1930	Schuman & Back	Flat Plates (Unstiffened)	Lateral Edges in Vee Grooves. Loaded Edges Milled Flat.	Plots of out-of-Plane Deflection vs. Load	First Use of Vee Grooves to Obtain Simple Support	Good
1930	Newell	Flat Plates	Stiffeners at Lateral Edges. Loaded Edges Milled Flat.	Visual Inspection	First Use of Edge Stiffeners to Prevent Edge Crippling	Good
1933	Cox	Flat Plates (Unstiffened)	Lateral Edges Bolted to Stirrups. Loaded Edges Clamped by Angles.	Change in Slope of Load vs. Strain Curve (Top of the Knee Method)	Attempt to Obtain Realistic Lateral Restraint, i.e., That Existing in a Wing	Good
1936	Gerard & Dickens	Curved Panels Stiffened by 2 Tubular Stringers	Lateral Edges Clamped in Wooden Side Members. Loaded Edges cast in a Low Melting Point Alloy.	Visual Inspection	The Use of a Low Melting Point Alloy to Obtain a Uniform Load Distribution	Good

Table 1 (Continued)

Date	Researcher	Specimen	Boundary Restraint	Method of Buckling Determination	Significant Features	Agreement With Theory
1938	Wenzek	Caisson of 3 Curved, Unstiffened Panels	Lateral Edges Simply Supported. Loaded Edges Milled Flat.	Visual Inspection	Use of a Caisson for Curved Panel Tests	Good
1939	Ramberg, McPherson & Levy	Flat, Stiffened Panels	Lateral Edges Held in Rounded Knife Edges. Loaded Edges Cast in Wood's Metal on Blocks Loaded Through Knife Edges.	Southwell Plot ( $P$ vs. $\delta/P$ ) from Optical Strain Gage Results & Photographs of Pointers Attached to the Specimen	Use of the Southwell Plot	Good
1940	Dunn	Flat, Stiffened Panels	Lateral Edges Held in Slotted Tubes - Loaded Edges Milled Flat.	Plotting the Load, $P$ , vs. the Buckle Amplitude Squared, $\delta^2$ , to Obtain $P_{crit}$	Development of the $P$ vs. $\delta^2$ Plot (intercept on $P$ -Axis is $P_{crit}$ )	Good
1941 (Published) (Done in 1934)	Cox & Clenshaw	Curved Panels (Unstiffened)	Lateral Edges Held in a "Comb" of Slotted Strips. Loaded Edges Clamped by Edges.	Plots of Load vs. Strain	Development of the "Comb" Assembly to Simulate Clamped or Fixed Lateral Edges	Fair

Table 1 (Continued)

Date	Researcher	Specimen	Boundary Restraint	Method of Buckling Determination	Significant Features	Agreement With Theory
1941	Lunquist	Curved and Flat Plates	Lateral Edges Stiffened by Angles-Loaded Edges Milled Flat.	Lunquist Plot for Flat & Slightly Curved Panels, Visual Inspection Otherwise	The First Use of Lunquist Plot in Plate Stability	Fair
1941	Holt	Flat, Stiffened Panels	Stiffeners at Lateral Edges. Loaded Edges Milled Flat, Saw Cut or on Knife Edges.	Visual Inspection	First Experimental Study to Vary Loaded Edge Conditions	Not Given
1944	Ramberg, Levy & Fienup	Curved, Stiffened Panels	Lateral Edges Held in Rounded Knife Edges. Loaded Edges Milled Flat.	Visual Inspection	The Use of Wire Resistance, SR-4, Strain Gages	Good
1945	Jackson & Hall	Curved & Flat, Unstiffened Panels	Lateral Edges Held in "Combs" or "Revised Combs." Loaded Edges Cast in Wood's Metal or Clamped by Blocks.	Measurement of Photographs of the Shadow Pattern Cast on the Panel During Testing by a Wire Grid	Development of the "Photogrammetric" Method	Fair



Table 1 (Continued)

Date	Researcher	Specimen	Boundary Restraint	Method of Buckling Determination	Significant Features	Agreement With Theory
1945- 1946	Welter	Curved & Flat Plates (Unstiffened)	Lateral Edges in Clamping "Combs"-Loaded Edges Clamped by Angles.	Load vs. Strain Diagram (Top of the Knee)	Studied Influence of "Cut outs" & Lateral Edge Stiffness. Showed Certain Imperfections to be Beneficial	Fair
1947	Farrar	Flat Plates	Lateral Edges Either Held by Rows of Ball or Roller Bearings. Loaded Edges Clamped by Angles.	Change in Slope of Load vs. Strain (Top of the Knee Method) and P vs. $\delta^2$ Plot (Strain vs. Buckle Amplitude Squared)	An Ambitious Lateral Edge Fixture	Not Good*
1947	Forest Products Laboratory	Flat Plates	Lateral Edges Held in Edge Guides or Clamped by Angles--Loaded Edges Held in Ball Bearings or Clamped by Angles.	Reversal of Strain Strain and Point of Inflection	First Use of Ball Bearing Assemblies to Obtain Simply Supported Loaded Edges	Excellent
1948	Hoff, Boley & Coan	Flat Plates	Lateral Edges Held in Knife Edges (Slotted) Tubes Discontinued)- Loaded Edges Held in Needle Bearings.	Load vs. Deflection, Load vs. Compressive Strain & Load vs. Bending Strain Diagrams	Strain Distribution could be Varied by Screws Between the Loading Block & the Needle Bearing Assemblies	Not Given

Table 1 (Concluded)

Date	Researcher	Specimen	Boundary Restraint	Method of Buckling Determination	Significant Features	Agreement With Theory
1959	Stein	Flat Plates	Lateral Edges in Knife Edges. Loaded Edges Milled Flat.	Strain Reversal	Multiple Knife Edge Sets to Create 11 Bays	Excellent
1967	Walker	Flat Plates	Lateral Edges in Knife Edges. Loaded Edges in Ball Bearings.	Southwell Plots	Experimental Investigation of Linearly Varying Axial Compression	Excellent
1969	Ashton & Love	Flat Plates	Lateral Edges in Knife Edges. Loaded Edges Clamped in Blocks.	Southwell Plots	Tests on Boron-Epoxy Composite Plates	Good
1971	Kicher & Mandell	Flat Plates	Lateral Edges in Ball Bearings. Loaded Edges Simply Supported.	Southwell Plots	The Use of Spring Loaded Pistons & Knife Edges for Uniform Compression	Good
1972	Tenerelli & Holmes	Curved Panels	Lateral Edges Held in Angles. Loaded Edges Potted in Epoxy.	Southwell Plots	A Loading Device Which Minimized Bending Moments at the Panel's Center	Excellent

\* On Farrar's own admission, his test system did not give the desired boundary effects.

## CHAPTER III

### ANALYTICAL STUDIES

The detailed review of panel experiments, given in Chapter II, shows that many research engineers have striven to obtain uniform load distributions and to achieve ideal boundary restraints. However, neither experimentalists nor theoreticians have investigated the influence of non-uniformity of either. Yet, in reality, some non-uniformity of loading and inequality in boundary restraints always occurs. The study presented in this chapter is therefore directed toward determining analytically the significance of inequality in edge rotational restraint parameters together with non-uniform loading. The importance of this analysis lies in the fact that its results should be of direct value to design engineers. Also, when combined with other studies being made of the vibration characteristics and the deformations of edge restrained plates, it might lead to a method for practically assessing the instability loads for realistic structures.

Problems of flat plates subject to an axial-compressive force whose intensity varies linearly across the width have been considered by an extremely small number of investigators. Timoshenko [50] gives buckling coefficients, at a limited number of aspect ratios, of simply supported plates subjected to each of the following loading conditions: a triangular distribution of axial compression, pure bending and one case of impure bending (greater compression than tension). Nokle [120] has

presented buckling coefficients for a plate with simply supported loaded edges and fixed (built-in) lateral edges both for pure bending and a triangular distribution of load. Both Timoshenko and Nokle utilized the energy method to obtain their solutions. Walker [111] used the Galerkin method in his solutions for the triangular and the trapezoidal distributions. An exact solution to the later case was developed by Hananel [121]. However, all of these investigators drastically curtailed the range of boundary conditions studied. This was due, of course, to the excessive amount of arithmetic which was involved with their methods of solution.

There would appear to be very little chance of improving this situation unless a judicious choice can be made of a function to use in a Rayleigh Quotient. By considering the physics of the problem, insight can be gained as to the possible nature of such a function. The plate under a triangular distribution of axial compression can be thought of as an assemblage of strips in the lengthwise direction (along the direction of load). This analogy, for example, has been applied with success by Zahorski [122] to the problem of a plate with transverse stiffening ribs and uniform axial compression. Each of these lengthwise strips behaves as a beam column under axial load. A normal force is developed which is proportional to the applied end load (axial compression). By this analogy, the normal load distribution which is developed across the width of the plate is of the same triangular distribution as the axial compression, and produces the out-of-plane deflections which occur across the width. Thus, the out-of-plane deflection pattern across the



plate's width can be approximated as the deflection of a beam with a triangular distribution of lateral load. For such an approximation to be reasonable, the boundary conditions at the ends of the beam must be analogous to those at the lateral edges of the plate in question.

Beam functions have been used for many years and by numerous investigators in the study of the vibration of plates. In his survey of such studies, Leissa [123] states that the first comprehensive collection of solutions for the vibrations of rectangular flat plates was presented by Warburton [124]. Warburton used the Rayleigh method with deflection functions which were the product of beam functions, i.e.,

$$W(x,y) = X(x)Y(y) \quad (6)$$

where  $X(x)$  and  $Y(y)$  were chosen as the fundamental mode shapes of beams having analogous boundary conditions to the plate. Such a technique was also used with success for various boundary conditions, by Young [125], Bazley [130], Fox and Stadter [126, 127, 128], Forsyth and Warburton [129], Barton [131], Lemke [132], Takahashi [133], and Carmichael [134]. Rhombic plates were analyzed in the same fashion by Kaul and Cadambe [135], Barton [136], and Claassen [137]. Other forms were investigated by Nagaraja et al. [138], Waller [139], and Kaul and Tewari [140].

The fact that such a technique yielded excellent results for vibrations is, in itself, motivation to apply the same method to plate buckling. Furthermore, there is also a useful analogy between the vibration and the buckling of flat rectangular plates which have two opposite simply supported sides. Leissa gives an excellent exposition of the

relationship which has been studied by Iguchi [141], Lurie [142] and Massonnet [143]. This analogy, which relates the buckling load and the natural frequency, indicates strongly that beam deflection functions may be employed in buckling studies in a similar manner as in vibrations.

#### The Simply Supported Plate Under Triangular Compression Load

The determination of the buckling coefficient of a flat plate with simply supported sides subjected to a triangular distribution of axial compression is a logical beginning. Since this case has been solved by Timoshenko and Gere [50] and by Walker [111], it affords a ready assessment of the accuracy achievable by using a beam deflection function. Intuitively, one would expect the deflection of a simply supported beam with a triangular lateral load to provide a good approximation to the chordwise deflection of the plate. The deflection formula for a simply supported beam with a triangular lateral load may be computed easily. It can be found in several reference texts and is given by Roark [144] as:

$$z = -\frac{1}{180} \frac{W y}{EI \ell^2} (3y^4 - 10\ell^2 y^2 + 7\ell^4) \quad (7)$$

for the configuration and coordinate system shown.

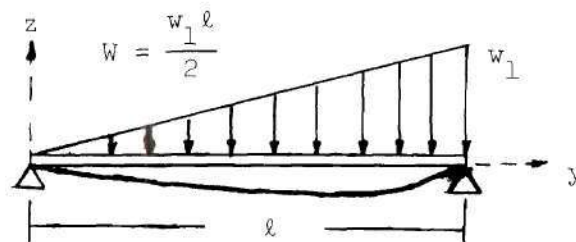


Figure 20. Beam With Triangular Lateral Load and Simply Supported Ends.

Since the loaded edges of the plate are simply supported, the deflection shape along the length can have a particularly convenient form, namely  $\sin \frac{n\pi x}{a}$  where "a" is the length of the plate.

Now the deflection function can be formed in the same manner as was done in the case of vibrations, i.e.,

$$w(x,y) = X(x)Y(y) \quad (6)$$

or

$$w(x,y) = \frac{1}{180} \frac{W}{EI} (10y^3 - 7b^2y - 3y^5/b^2) \sin \frac{n\pi x}{a} \quad (8)$$

where  $b$  = the width of the plate

$W$  = the load.

Since the term  $(1/180)(W/EI)$  is a constant, it will be replaced by the constant  $\lambda/3$ , where  $\lambda = (1/60)(W/EI)$ . Hence, the deflection function may be written as

$$w(x,y) = (\lambda/3)(10y^3 - 7b^2y - 3y^5/b^2) \sin \frac{n\pi x}{a} \quad (9)$$

Following Raleigh, a criterion for the buckling of a plate is the satisfaction of the equality which result from making the work done by the external forces balance the strain energy of plate bending. The pertinent equation is given by Timoshenko as

$$\begin{aligned} \frac{1}{2} \iint N_x \left( \frac{\partial w}{\partial x} \right)^2 dx dy - \frac{D}{2} \iint \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 \right. \\ \left. - 2(1 - \nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy = 0 \end{aligned} \quad (10)$$

Now  $N_x$  may be expressed as

$$N_x = N_o [(1 - \alpha) + \alpha y/b] \quad (11)$$

where  $\alpha = 0$  for an uniform load distribution

$\alpha = 1/2$  for a trapezoidal load distribution

$\alpha = 1$  for a triangular load distribution.

Thus, the Rayleigh Quotient becomes:

$$N_o = \frac{D \int_0^b \int_0^a \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy}{\int_0^b \int_0^a [(1 - \alpha) + \alpha y/b] \left( \frac{\partial w}{\partial x} \right)^2 dx dy} \quad (12)$$

Substitution of the deflection function and performance of the indicated operations will yield values for the critical buckling coefficient  $K$  ( $K = -N_o b^2/D$ ) as a function of  $a/b$  (the length-to-width ratio). These values are plotted in Figure 21 where they are compared with the values given by Walker [111]. It can be seen that the results of the present analysis are in excellent agreement with the prior work.

This method of utilizing an appropriate beam deflection function and a Rayleigh Quotient can also be applied to the case of a plate with one lateral edge simply supported and the other lateral edge fixed (built-in). In this case, the loaded edges will remain simply supported and the triangular distribution of axial compression will have its peak value at the fixed edge, consequently the compression will go to zero at the simply supported edge. The corresponding beam deflection function for this case is given by Roark [144] as



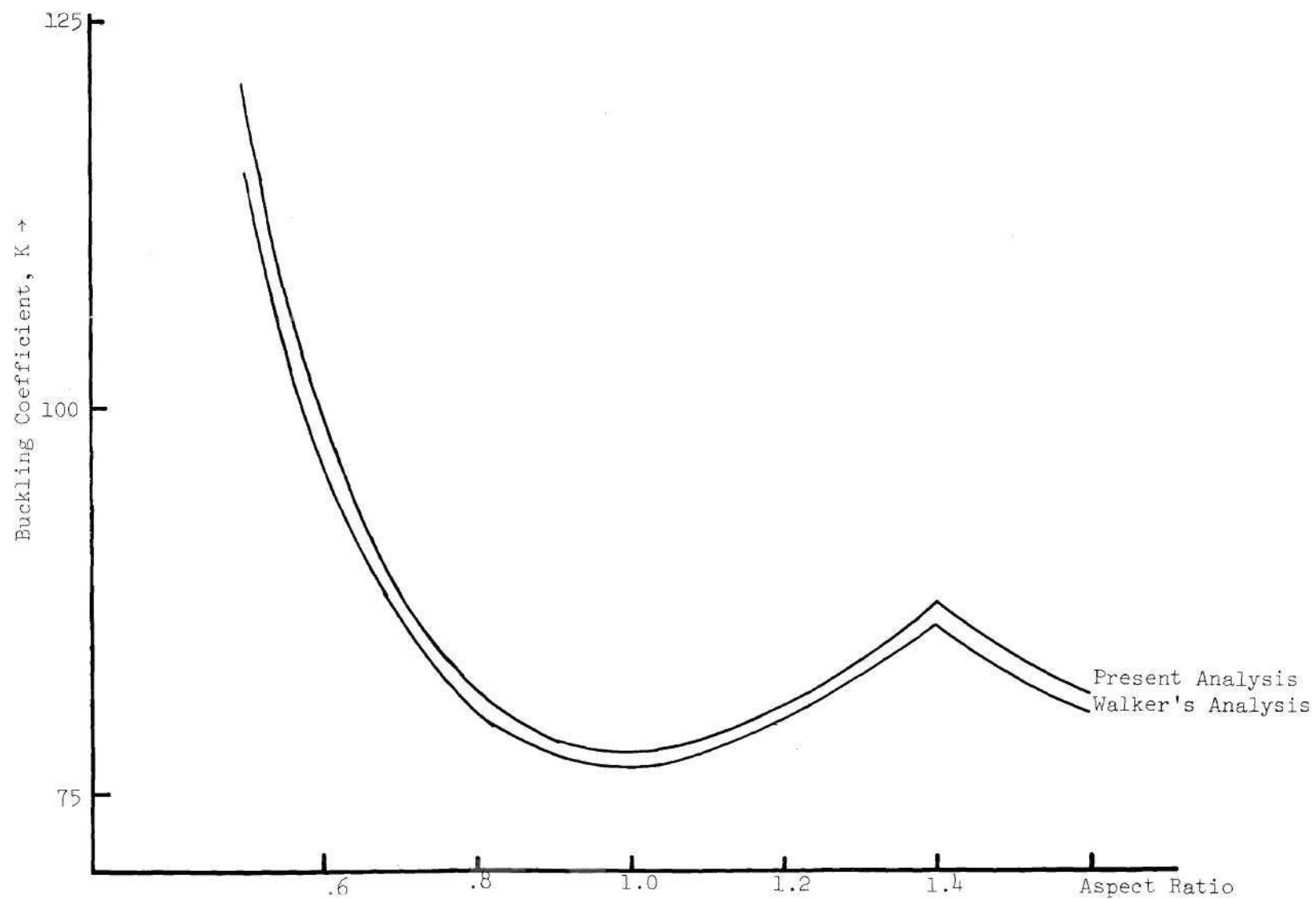


Figure 21. Buckling Coefficients Triangular Load Simply Supported Lateral Edges.

$$z = \frac{1}{60} \frac{W}{EI\ell} (2\ell y^3 - \ell^3 y - y^5/\ell) \quad (13)$$

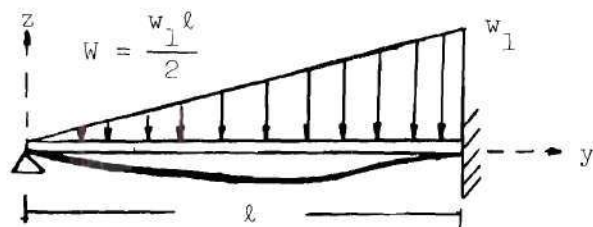


Figure 22. Beam With Triangular Lateral Load and Simply Supported-Fixed Ends.

Following the same procedure as in the previous case, the deflection function for this plate may be written as

$$w(x,y) = \lambda (2y^3 - b^2 y - y^5/b^2) \sin \frac{n\pi x}{a} \quad (14)$$

Again substitution into and evaluation of the Rayleigh Quotient, as before, will yield values of the buckling coefficient  $K$  in terms of the aspect ratio,  $a/b$ , of the plate. These values are plotted in Figure 23 where they may be compared with results given by Walker [111] for this case. The apparent discrepancy between the two curves merits discussion. In the text of his paper, Walker illustrates the convergence of his calculation scheme by citing as an example this plate case with an aspect ratio of one. The pertinent portion of his text is reproduced as this juncture.

- (ii) Eccentrically-compressed ( $\alpha = 1$ ) square plate, simply-supported along one edge, built-in along the other ( $K_1 = 0$ ;  $K_0 = \infty$ ):

<u>Terms included</u>	<u>Eigenvalue</u>
$q_0$	130.8019
$q_0 \ q_1$	125.2303
$q_0 \ q_1 \ q_2$	125.2001
$q_0 \ q_1 \ q_2 \ q_3$	125.1820
$q_0 \ q_1 \ q_2 \ q_3 \ q_4$	125.1800
$q_0 \ q_1 \ q_2 \ q_3 \ q_4 \ q_5$	125.1797
$q_0 \ q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6$	125.1797

The value obtained by Walker, i.e.,  $K = 125.1797$ , is virtually identical to that of the present analysis for  $a/b = 1$  as is shown in Figure 23. The convergence of his calculation scheme to this eigenvalue for the square plate is strong evidence that his solution method is correct for the other aspect ratios. The anomaly between calculation and curve for the square plate is explicable as an incorrect plotting of the calculated values.

#### The Fixed-Fixed Plate Under Triangular Load

The next case to be considered is that of a plate under triangular axial compression with its lateral edges fixed (built-in) and its loaded edges simply supported. The beam deflection function applicable to this case, as given by Roark [144], is

$$z = \frac{1}{60} \frac{W}{EI} (3y^3 - 2ly^2 - y^5/l^2) \quad (15)$$

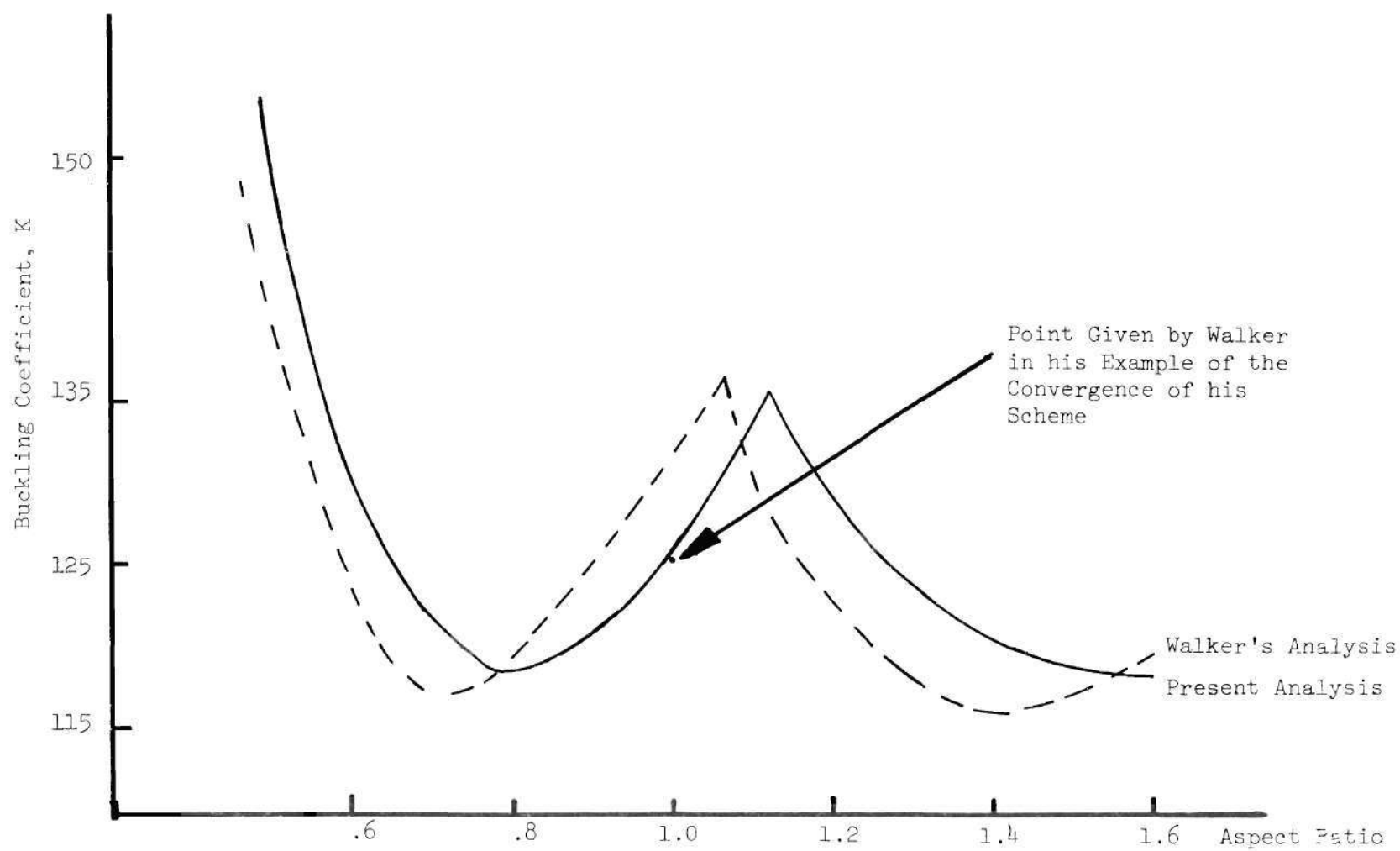


Figure 23. Buckling Coefficients Triangular Load Simply Supported-Fixed Lateral Edges.



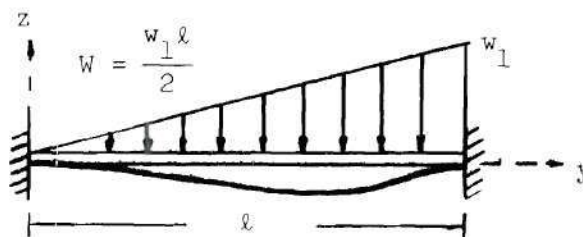


Figure 24. Beam With Triangular Lateral Load and Fixed Ends.

The deflection function then is expressed as

$$w(x,y) = \lambda(3y^3 - 2by^2 - y^5/b^2) \sin \frac{n\pi x}{a} \quad (16)$$

The substitution into the Rayleigh Quotient follows as before and values of  $K$  versus aspect ratio are yielded. These are plotted in Figure 25 as are the values given by Timoshenko [50] (from the work of Nolke). Again good agreement is achieved. As far as the present author has been able to trace, these are the only solutions for this particular loading and boundary condition.

#### The Plate With Equal Elastically Restrained $(\beta, \beta)$

##### Lateral Edges Under Triangular Load

After having achieved a solution for the plate with lateral edges fixed and for the plate with lateral edges simply supported, the next logical step is to consider the plate which has equal rotational restraints at its lateral edges. Such a plate has the two aforementioned cases as its extremes and hence the rectitude of the solution will be readily apparent. In such a study, guidance may be obtained from the

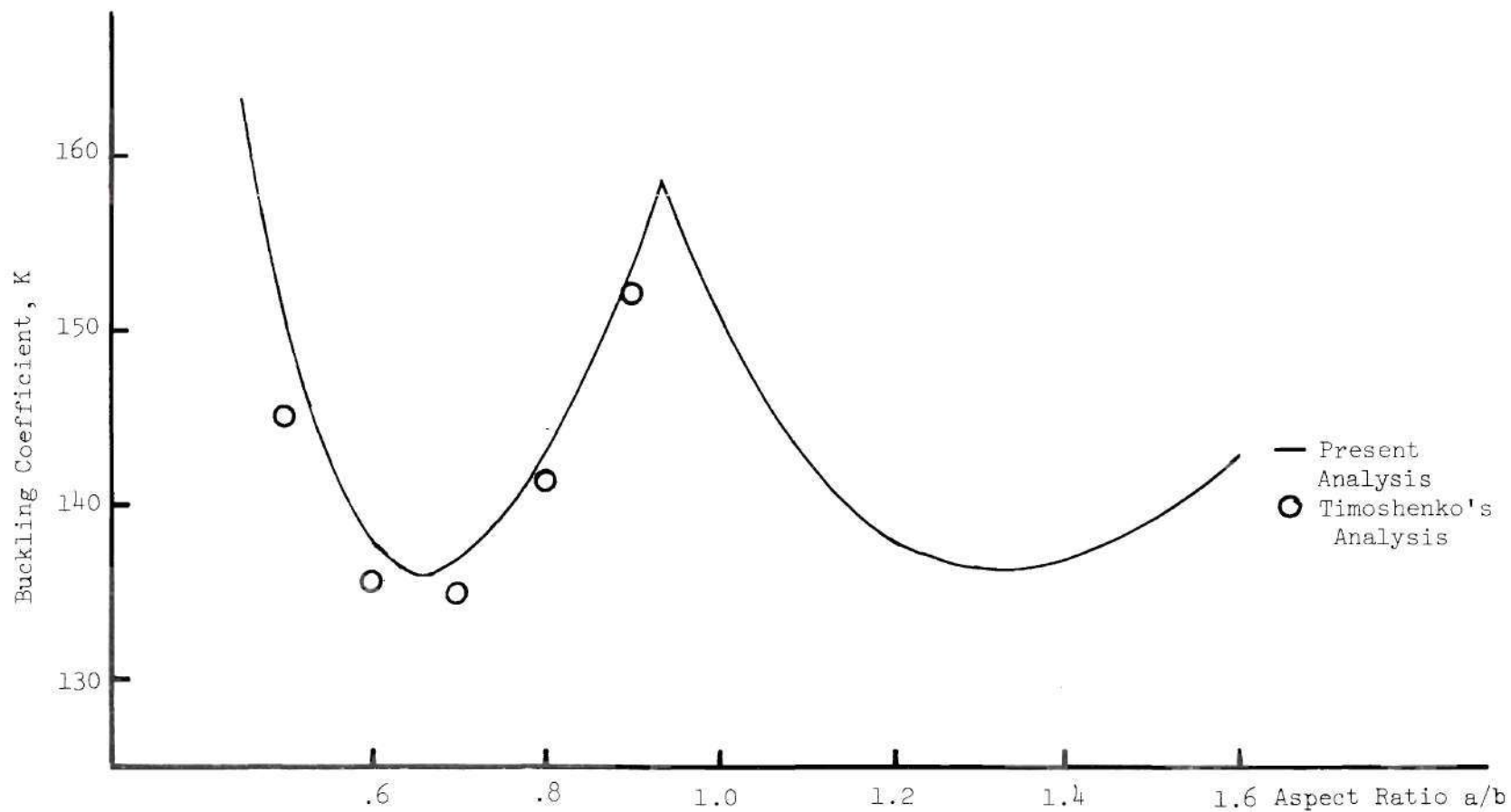


Figure 25. Buckling Coefficients Triangular Load Fixed-Fixed Lateral Edges.

work of Lundquist and Stowell [145] who analyzed a plate under uniform axial compression with equal lateral rotational restraints. The corresponding computations for a triangular distribution of axial compression do not appear to have been done previously.

In their work, Lundquist and Stowell began by considering that a plate with no elastic restraints at its edges would buckle in a sine curve while a plate with equal elastic restraints at its edges would deflect into a circular arc. Thus, they selected as their deflection curve across the width of the plate, a curve given by the sum of a circular arc and a sine curve. Hence, their deflection curve was

$$w(x,y) = \left[ \frac{4A^*}{b^2} (y^2 - b^2/4) + \left( \frac{4A^*}{\pi} + B^* \right) \cos \frac{\pi y}{b} \right] \sin \frac{\pi x}{\lambda} \quad (17)$$

where  $A^*$  and  $B^*$  are "arbitrary deflection amplitudes" such that for  $A^* = 0$  the plate is simply supported at the lateral edges and for  $B^* = 0$  the plate is fixed at the lateral edges.

Now, as Lundquist and Stowell state, the ratio of  $A^*/B^*$  is a measure of edge restraint and is related to the restraint coefficient through the boundary condition

$$-D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \bigg|_{y=b} = 4S_0 \left( \frac{\partial w}{\partial y} \right) \bigg|_{y=b} \quad (18)$$

and by definition,

$$\epsilon = \frac{4S_0 b}{D} \quad (19)$$

where  $S_0$  = stiffness per unit length of elastic restraining medium  
or moment required to rotate a unit length of elastic  
medium through one-fourth radian

$b$  = the plate width

$D = (Et^3)/(12(1-\nu^2))$ , the plate flexural rigidity.

Using this boundary condition,  $A^*$  may be expressed in terms of  $B^*$  as follows,

$$A^* = .3927\epsilon B^* . \quad (20)$$

Thus the Rayleigh Quotient contains terms which express the strain energy of the elastic restraining mediums at the two edges of the plate. Hence, it is of the form:

$$\begin{aligned} N = & \left\{ D \int_0^a \int_0^b \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \right. \\ & + 4S_0 \int_0^a \left[ \left( \frac{\partial w}{\partial y} \right) \Big|_{y=0} \right]^2 dx + 4S_0 \int_0^a \left[ \left( \frac{\partial w}{\partial y} \right) \Big|_{y=b} \right]^2 dx \left. \right\} / \int_0^a \int_0^b \left( \frac{\partial w}{\partial x} \right)^2 dy dx . \end{aligned} \quad (21)$$

Values of  $K$  can be obtained from this equation for various values of  $\epsilon$  and  $a/b$ .

A similar analysis can be conducted for the case of a triangular distribution of axial compression acting upon a plate with equal elastic lateral restraints. In this case, the deflection function will be formed as a weighted mean of the simply supported and the fixed beam deflection functions. If  $A^*$  and  $B^*$  are the weighting parameters then,



$$w(x,y) = \{B^*(10y^3 - 7b^2y - 3y^5/b^2) + A^*(3y^3 - 2by^2 - y^5/b^2)\} \sin \frac{n\pi x}{a} \quad (22)$$

where  $A^* = 0$  yields simple support and

$B^* = 0$  yields fixed lateral edges.

Utilizing the same boundary condition as before, namely

$$-D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x \partial y} \right) \bigg|_{y=b} = 4S_0 \left( \frac{\partial w}{\partial y} \right) \bigg|_{y=b} \quad (23)$$

the following relationship is obtained between  $A^*$  and  $B^*$ ,

$$A^* = .444\epsilon B^* \quad (24)$$

Hence, the deflection function for the triangular distribution of axial compression acting on a plate with equal lateral elastic restraints may be written as

$$w(x,y) = B^* \left\{ (4\epsilon + 10)y^3 - \left( \frac{4}{3}\epsilon + 3 \right) y^5/b^2 - \left( \frac{8}{3}\epsilon by^2 - 7b^2y \right) \sin \frac{n\pi x}{a} \right\} \quad (25)$$

The Rayleigh Quotient then becomes

$$N = \left\{ D \int_0^a \int_0^b \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \right. \\ \left. + 4S_0 \int_0^a \left[ \left( \frac{\partial w}{\partial y} \right) \bigg|_{y=0} \right]^2 dx + 4S_0 \int_0^a \left[ \left( \frac{\partial w}{\partial y} \right) \bigg|_{y=b} \right]^2 dx \right\} / \int_0^a \int_0^b$$

$$[(1 - \alpha) + \alpha y/b](\frac{\partial w}{\partial x})^2 dydx \quad (26)$$

The results of this analysis for various values of the edge restraint coefficient and aspect ratio are shown in Figure 26.

#### The Fixed-Simply Supported Plate Under Triangular Load

There is another case to which no prior solution has been given. It is the case of a triangular distribution where the peak load acts at a simply supported edge while the zero load edge is fixed. A beam deflection function is available for this case and its employment yields the following deflection function for the plate in question, viz.

$$w(x,y) = (\lambda/2)(9y^3 - 7by^2 - 2y^5/b^2) \sin \frac{n\pi x}{a} \quad (27)$$

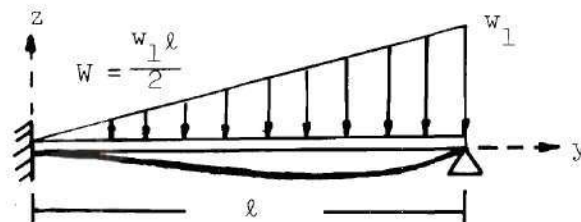


Figure 27. Beam With Triangular Lateral Load and Fixed-Simply Supported Ends.

The values of  $K$  versus aspect ratio for this case are given in Figure 28. The reader may refer back to Figure 23 for the case when the lateral boundary conditions were reversed, i.e., the peak load at the fixed edge and zero load at the simply-supported edge.

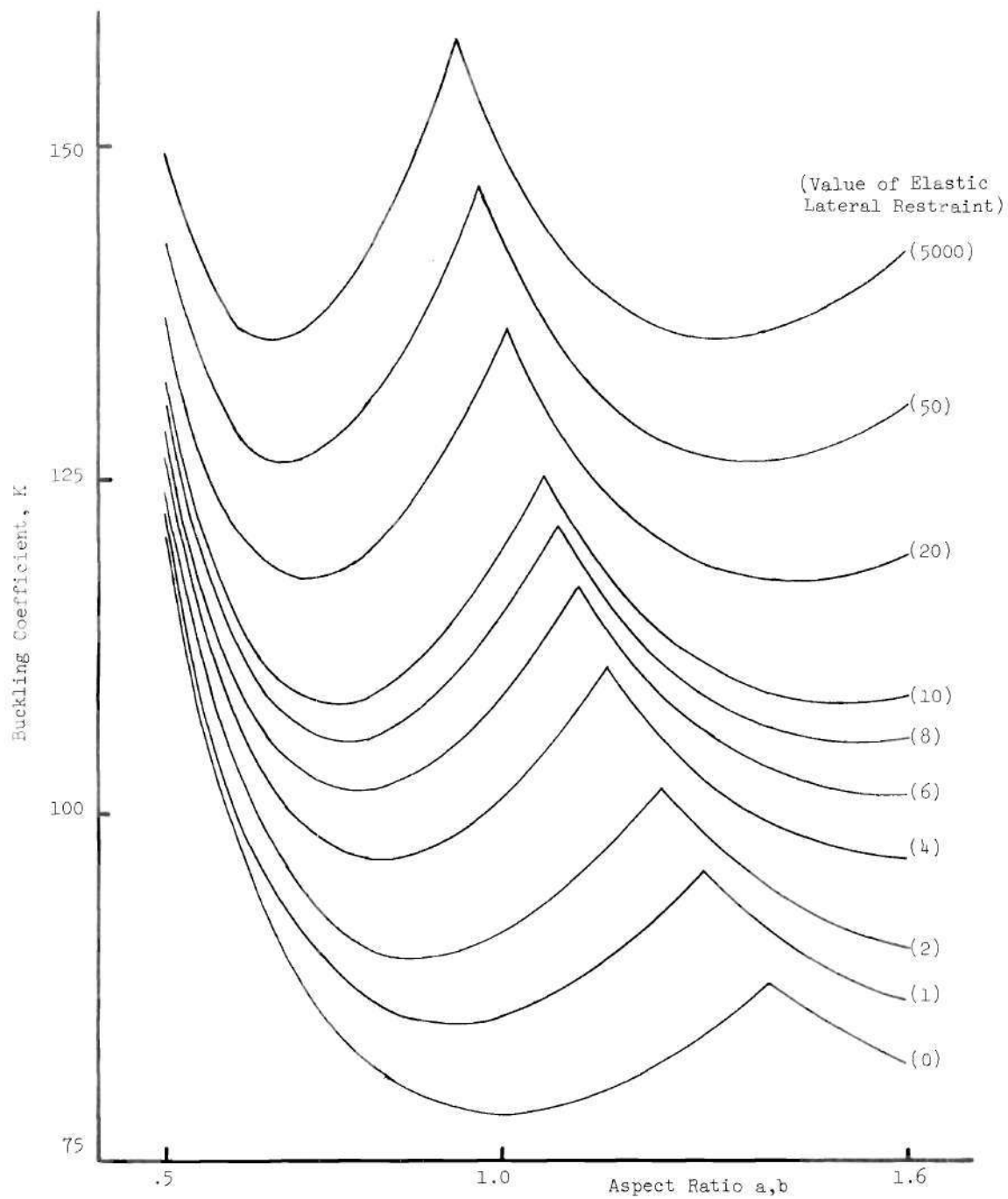


Figure 26. Buckling Coefficients Triangular Load ( $\beta, \beta$ ) Lateral Edges.

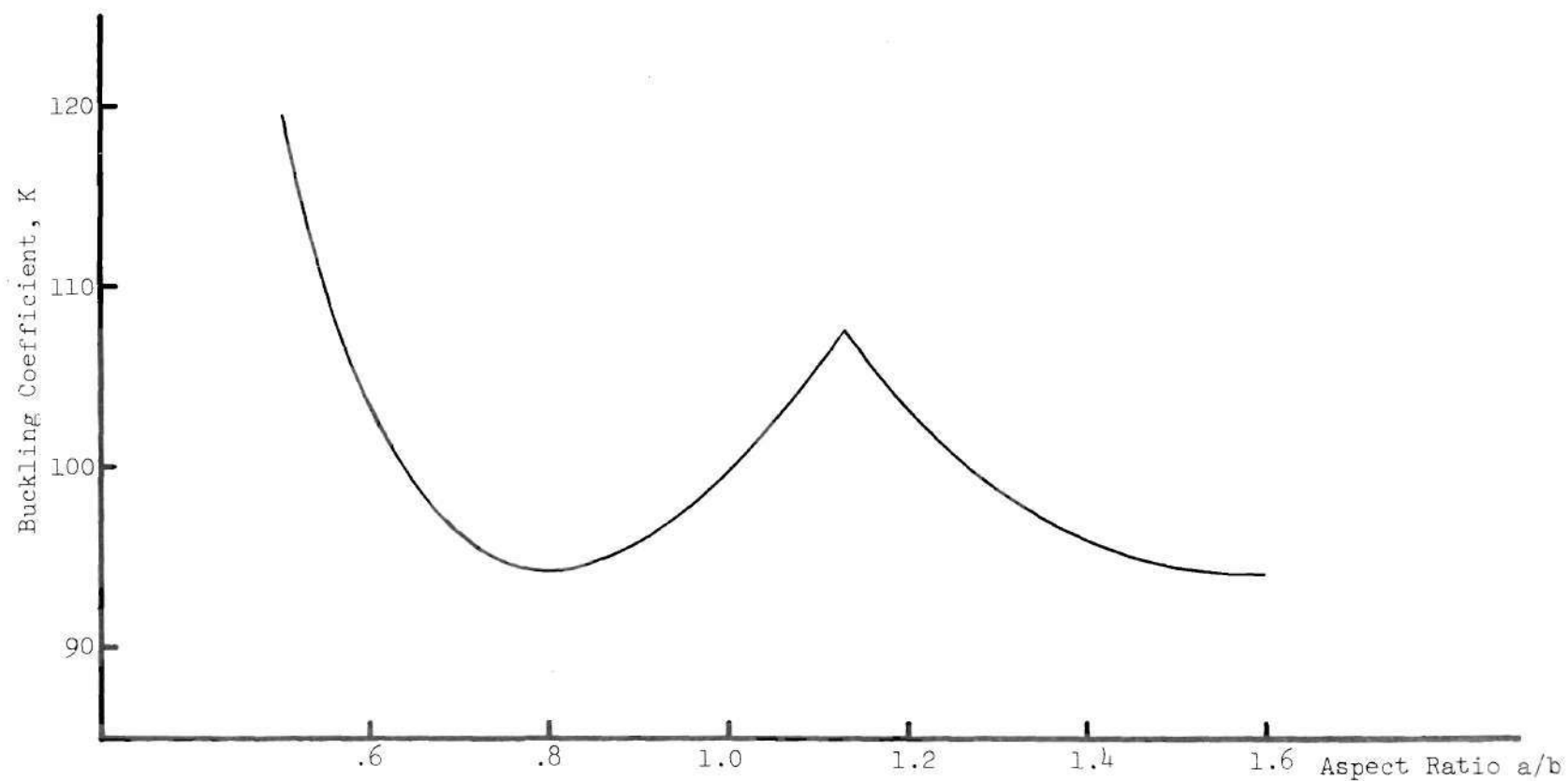


Figure 28. Buckling Coefficients Triangular Load Fixed-Simply Supported Lateral Edges.



The Plate With Unequal Elastically Restrained ( $\alpha, \beta$ )  
Lateral Edges Under Triangular Load

The case now to be analyzed is one of more than academic interest because equality of edge restraint parameter is not likely to be achieved in practice. Nassar [146] has shown that for any lateral load distribution the deflection of a beam with unequal elastic end restraint parameters,  $\beta_1$  and  $\beta_2$ , can be expressed in terms of these parameters and the deflection functions for the several extreme restraint conditions.

The general expression is

$$z = \frac{\beta_1 \beta_2 z_{\infty\infty} + 4\beta_1 z_{\infty 0} + 4\beta_2 z_{0\infty} + 12z_{00}}{\beta_1 \beta_2 + 4\beta_1 + 4\beta_2 + 12} \quad (28)$$

where  $\beta_1$  and  $\beta_2$  are the elastic restraint parameters and the  $z$ 's the various deflection functions. The suffixes  $\beta_1 \beta_2$ ,  $\infty\infty$ ,  $00$ , etc., define the pertinent end fixity condition. Thus,  $\infty\infty$  implies both ends fixed and  $00$  both ends simply supported. In general, the leading suffix implies condition at the left hand end of beam and the trailing suffix those at the right. Thus, in accord with the previously adopted procedure, the plate deflection function will be written as

$$w(n, y)_{\beta_1 \beta_2} = \left\{ \frac{\beta_1 \beta_2 z_{\infty\infty} + 4\beta_1 z_{\infty 0} + 4\beta_2 z_{0\infty} + 12z_{00}}{\beta_1 \beta_2 + 4\beta_1 + 4\beta_2 + 12} \right\} \sin \frac{n\pi x}{a} \quad (29)$$

The appropriate extreme restraint deflection functions are

$$z_{\infty\infty} = \lambda(3y^3 - 2by^2 - y^5/b^2) \quad (30)$$

the deflection function of a beam under a  
 triangular lateral load with both ends fixed

$$z_{0\infty} = \lambda(2y^3 - by^2 - y^5/b^2) \quad (31)$$

the deflection function of a beam under a triangular lateral load with one end simply supported (end with zero load) and the other end fixed (end with peak load).

$$z_{\infty 0} = (\lambda/2)(9y^3 - 7by^2 - 2y^5/b^2) \quad (32)$$

the deflection function of a beam under a triangular lateral load with one end fixed (end with zero load) and the other end simply supported (end with peak load)

$$z_{00} = (\lambda/3)(10y^3 - 7by^2 - 3y^5/b^2) \quad (33)$$

the deflection function of a simply supported beam under a triangular lateral load.

It follows then that when the various terms are grouped according to powers of  $y$  the displacement function becomes

$$\begin{aligned} w(x,y) = \lambda \left( \frac{1}{\beta_1\beta_2 + 4\beta_1 + 4\beta_2 + 12} \right) & \left[ (3\beta_1\beta_2 + 8\beta_2 + 18\beta_1 + 40)y^3 \right. \\ & - (2\beta_1\beta_2 + 14)by^2 - (4\beta_2 + 28)b^2y \\ & \left. - (\beta_1\beta_2 + 4\beta_1 + 4\beta_2 + 12)y^5/b^2 \right] \sin \frac{n\pi x}{a} . \end{aligned} \quad (34)$$

Now the term  $\{\lambda/(\beta_1\beta_2 + 4\beta_1 + 4\beta_2 + 12)\}$  simply defines the attenuation factor which insures a proper value for the amplitude of the deflection. Thus, it does not enter into the Rayleigh Quotient and we may conveniently consider it constant. The results portrayed in Figures 29

through 34 illustrate the power of the approach.

In the studies which have been made so far the pertinent deflection functions have been selected in accordance with a specific procedure, and have thus had a uniqueness. It seems reasonable to conjecture, however, that there may well be considerable latitude in the function. Temple and Bickley [147] give some indication of this possibility. They state; "In a beam, for instance, a variation of the loading produces a relatively much smaller variation in the deflection; so that the approximation to the deflexion obtained from an approximation to the loading, is proportionally closer to the true shape than the loading is to the true loading."

To pursue this point further, the plate behavior under a trapezoidal distribution of axial compression was investigated using the displacement function chosen for a triangular load distribution. For a plate with 0 and  $\beta$  lateral edge restraint parameters, excellent agreement was obtained with the results derived by Walker [111].

In view of this most satisfactory correspondence, the more general case of  $\alpha, \beta$  lateral edge restraint and a trapezoidal loading distribution was determined using the same procedure. The results are shown in Figures 35 through 41.

To strengthen our confidence in the validity of these results, the case of a uniform distribution was evaluated using a displacement function pertinent to the triangular loading case. This case is well documented in the literature for the conditions of equal restraint along the lateral edges, e.g., the work of Lundquist and Stowell [145].

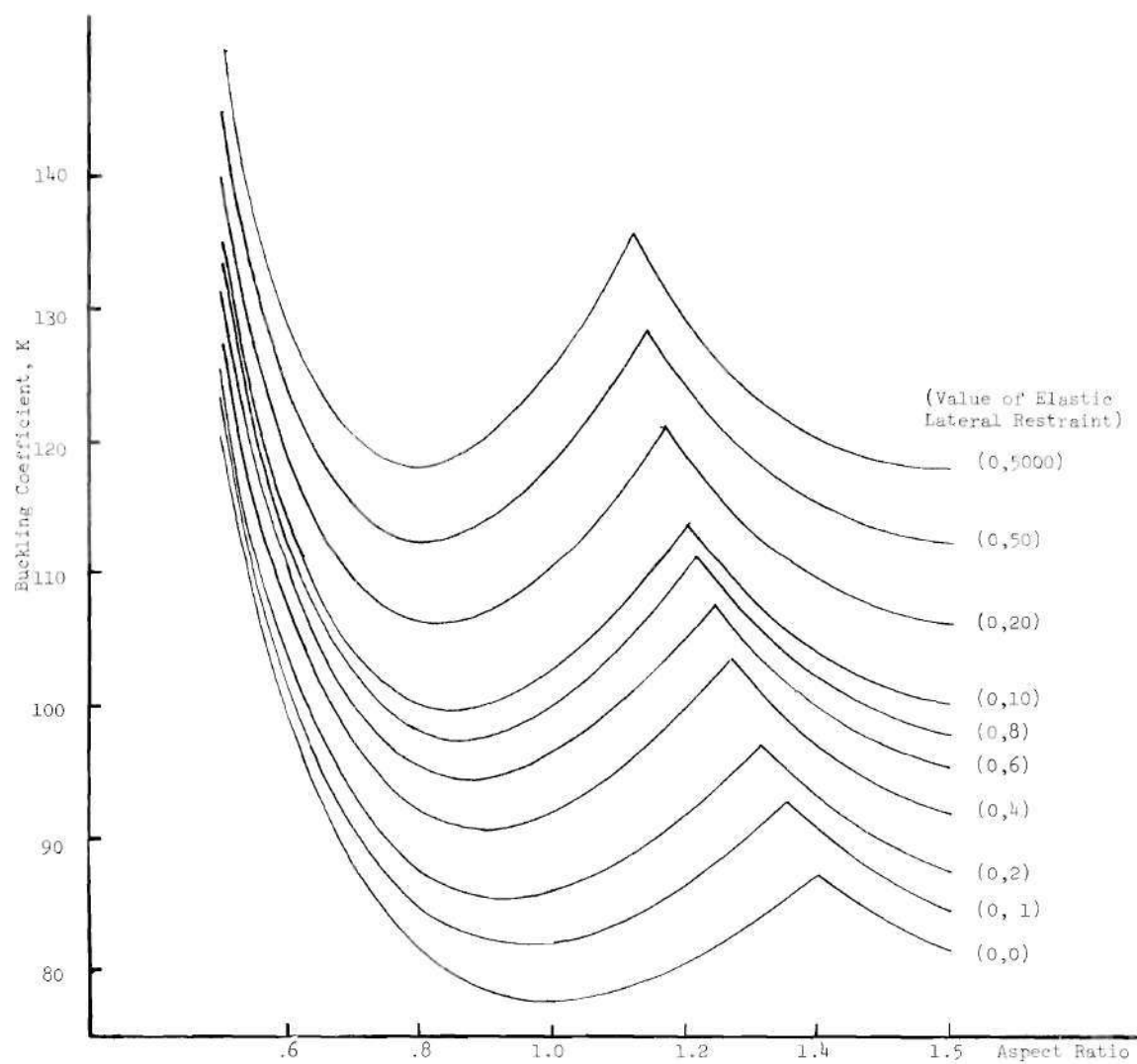


Figure 29. Buckling Coefficients Triangular Load (0,6) Lateral Edges.



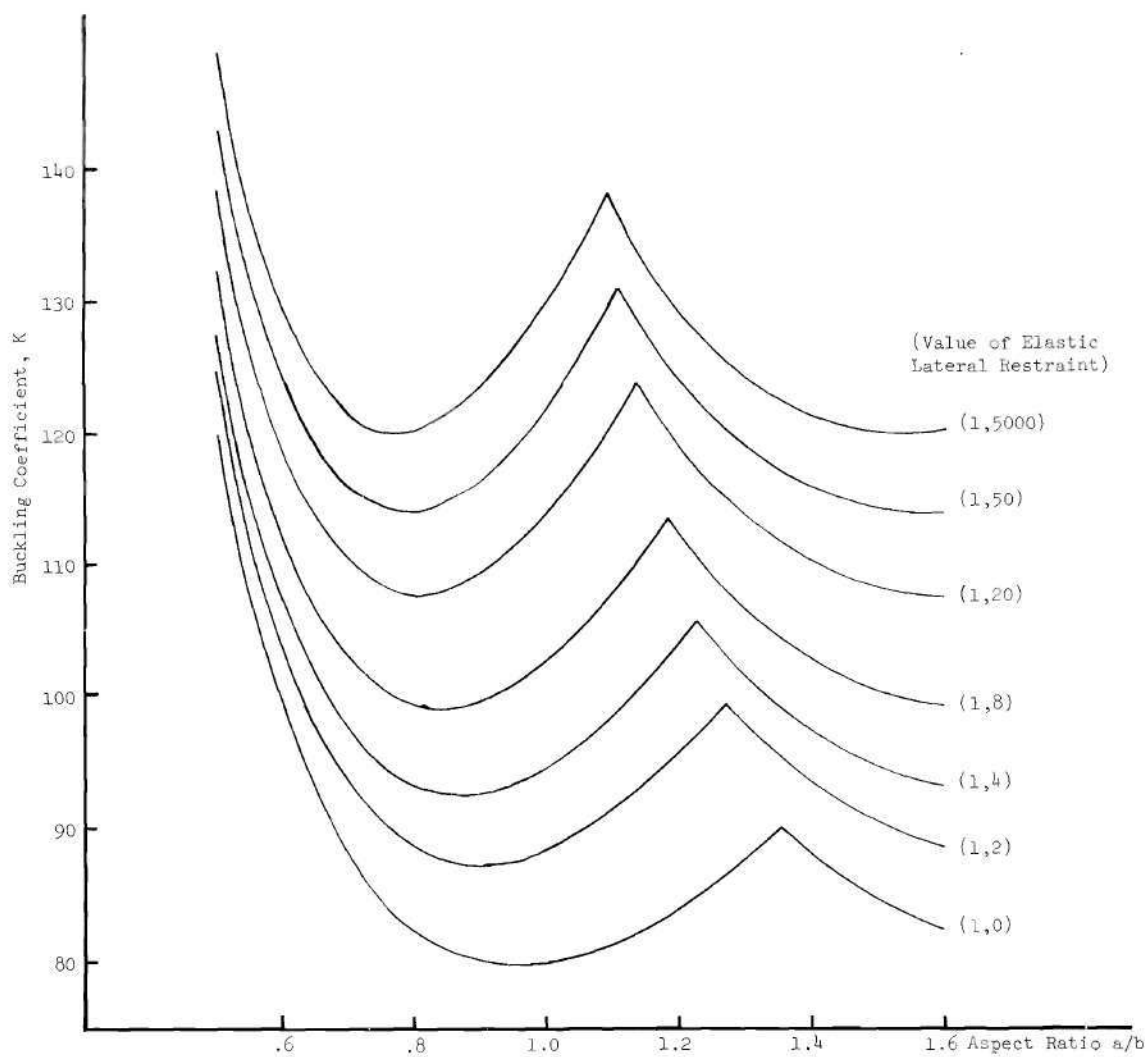


Figure 30. Buckling Coefficients Triangular Load (1,8) Lateral Edges.

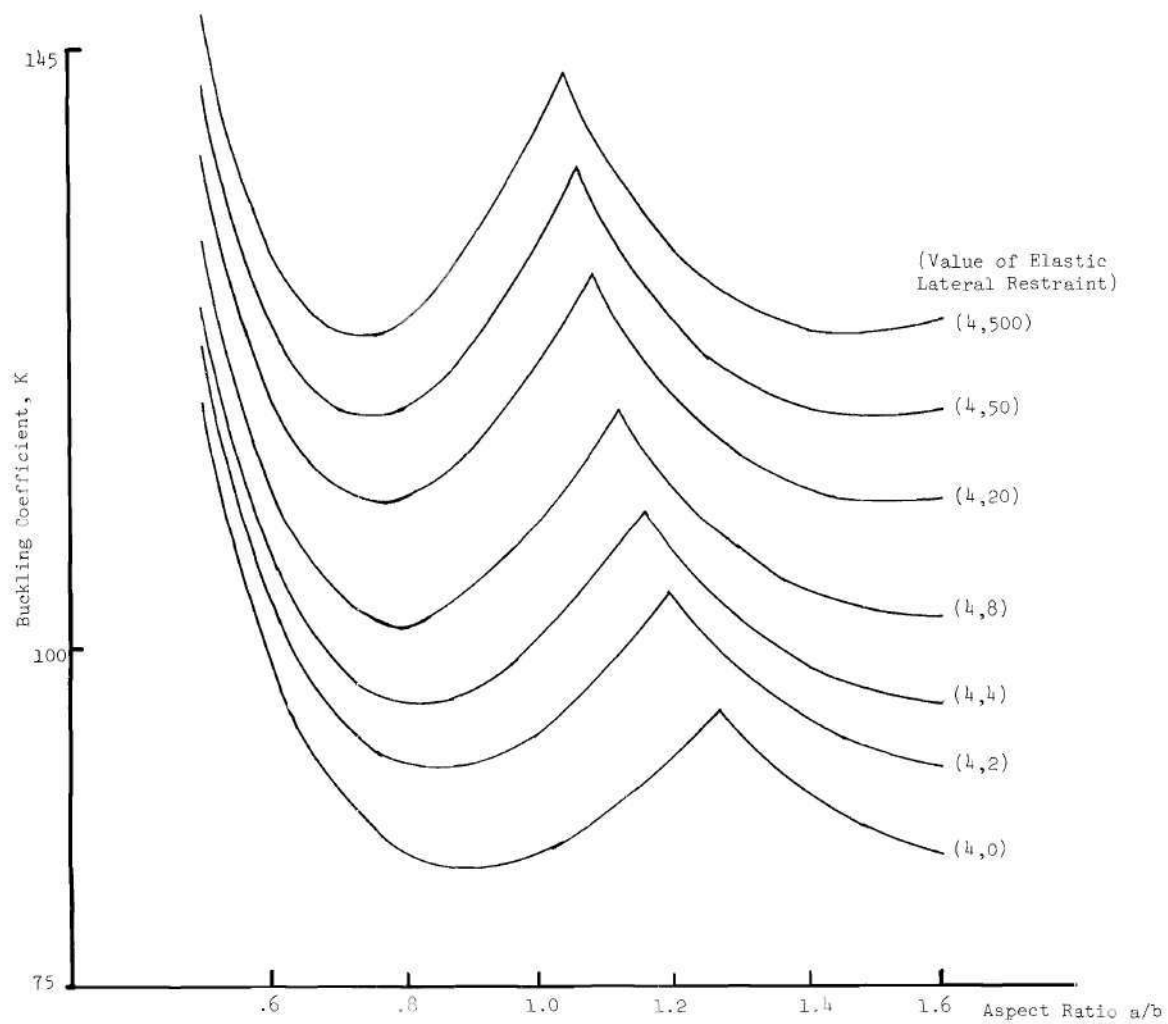


Figure 31. Buckling Coefficients Triangular Load  $(4, \beta)$  Lateral Edges.

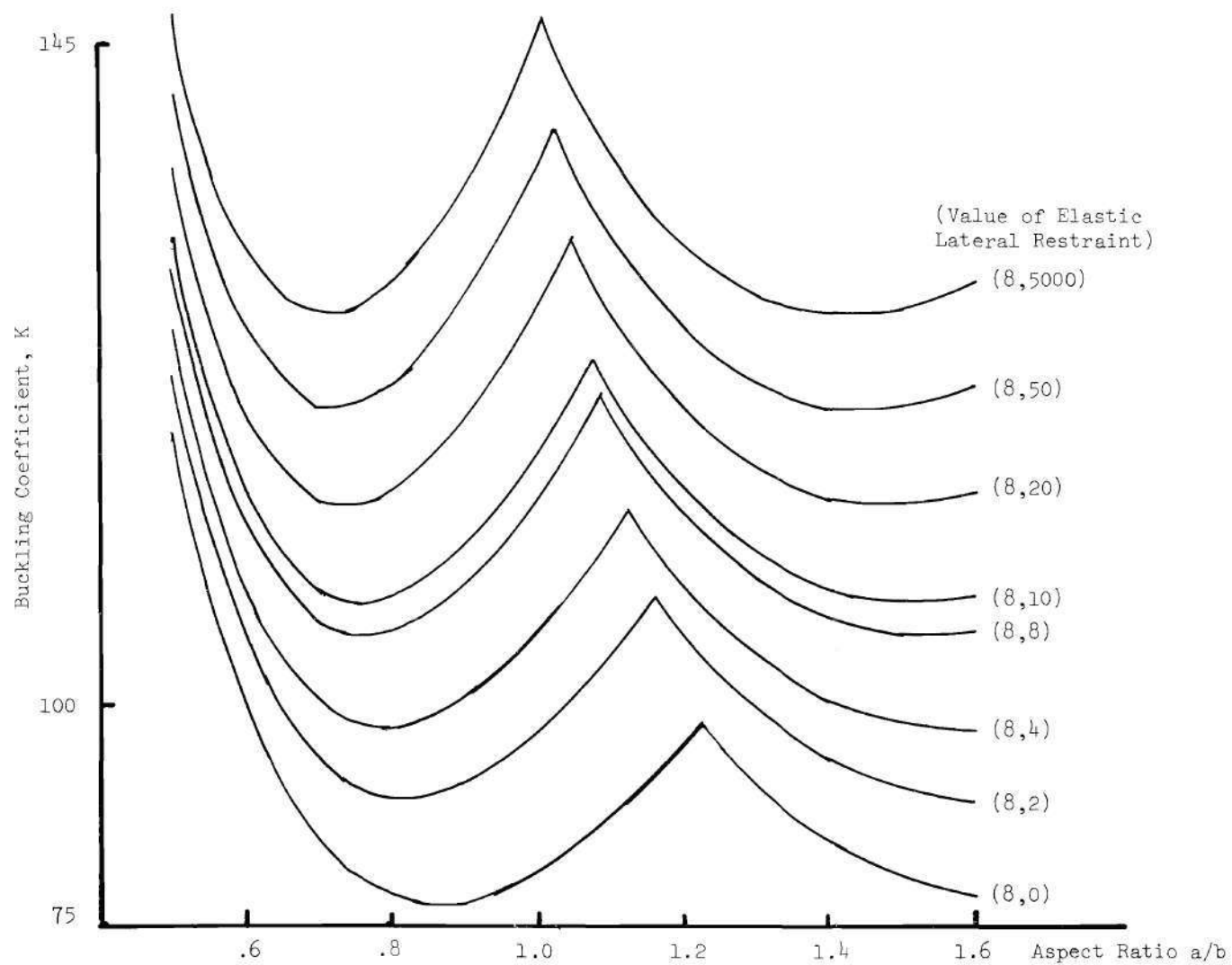


Figure 32. Buckling Coefficients Triangular Load  $(8, \beta)$  Lateral Edges.

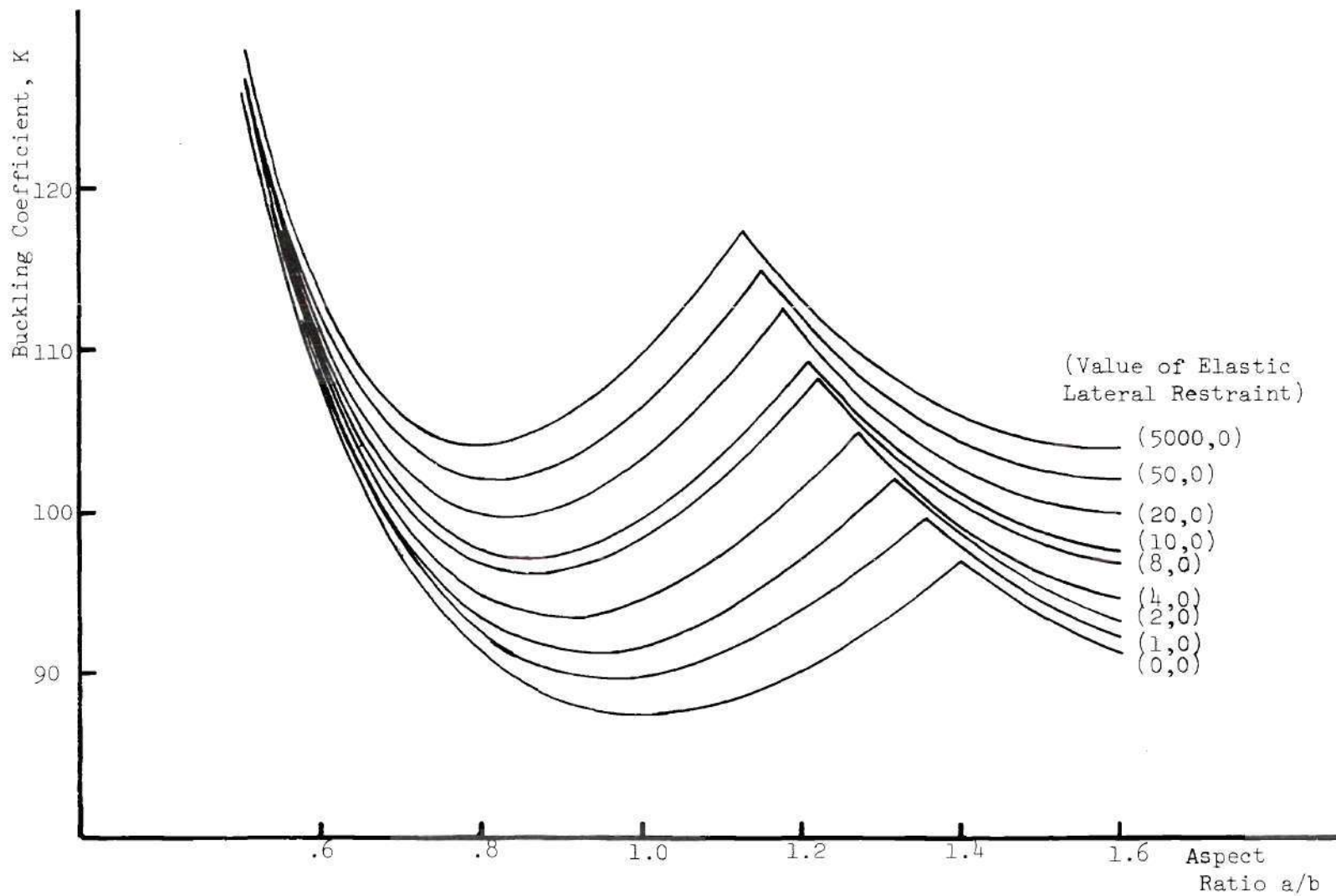


Figure 33. Buckling Coefficients Triangular Load  $(\alpha, 0)$  Lateral Edges.



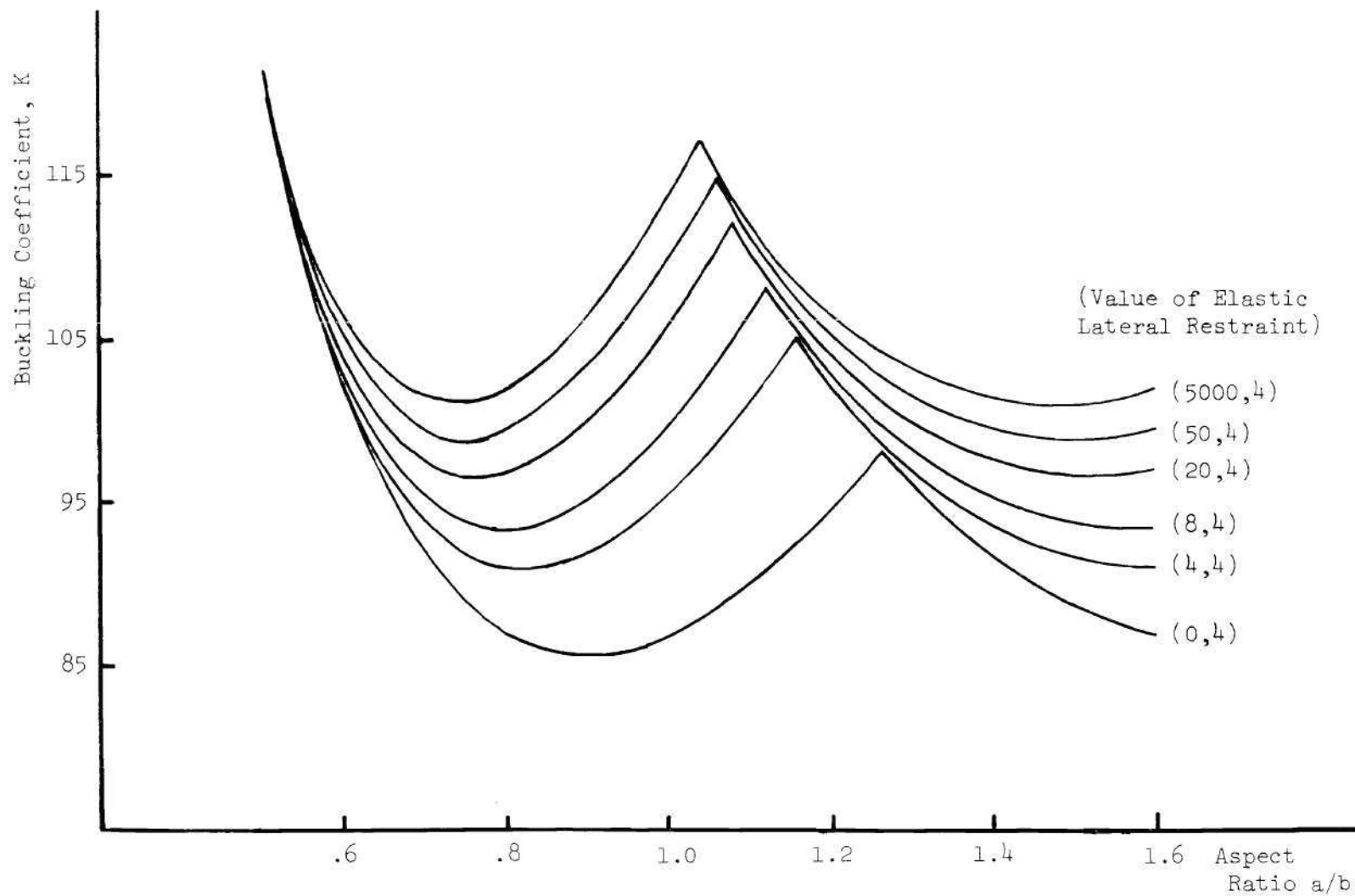


Figure 34. Buckling Coefficients Triangular Load ( $\alpha, 4$ ) Lateral Edges.

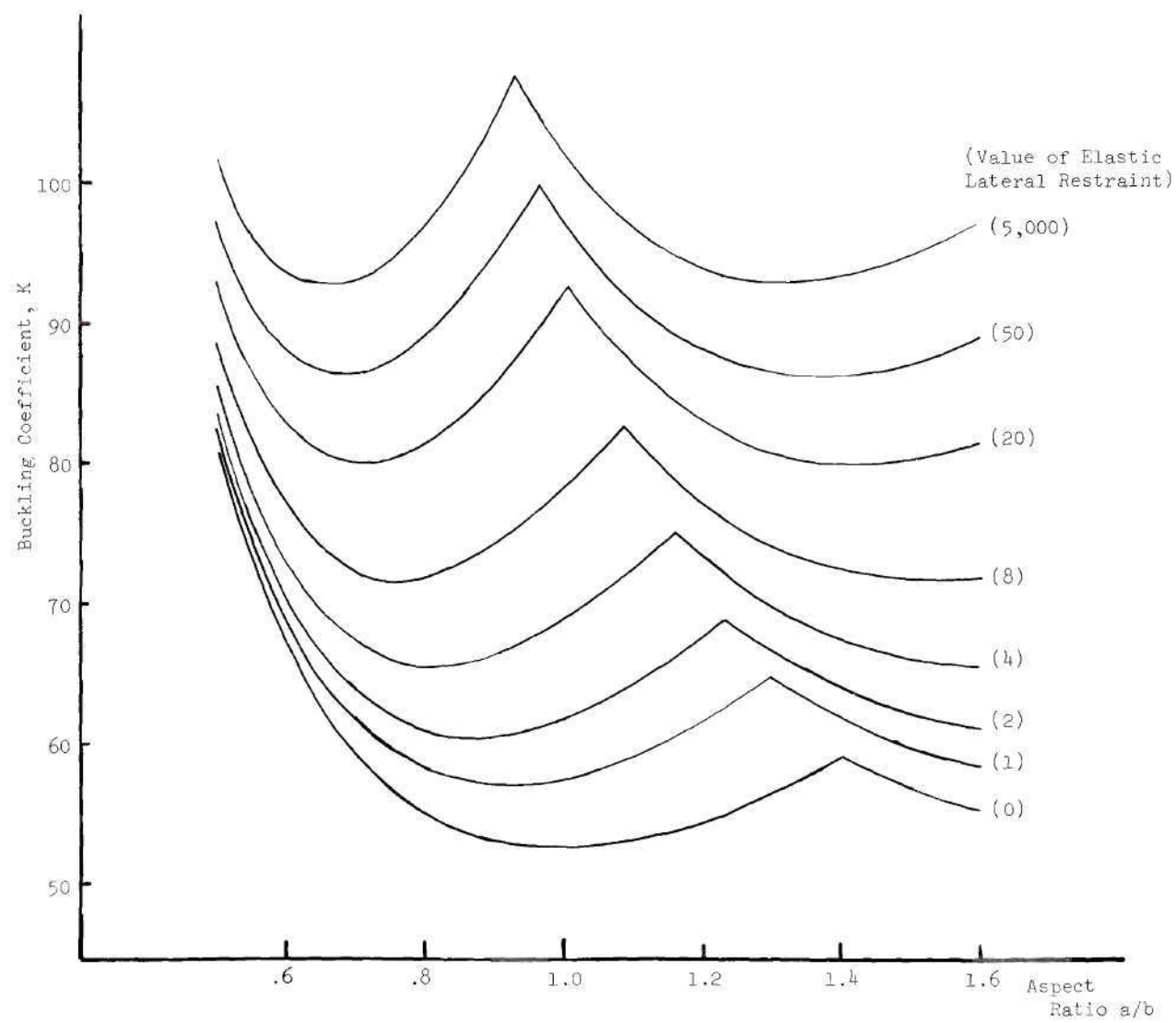


Figure 35. Buckling Coefficients Trapezoidal Load ( $\beta, \beta$ ) Lateral Edges.

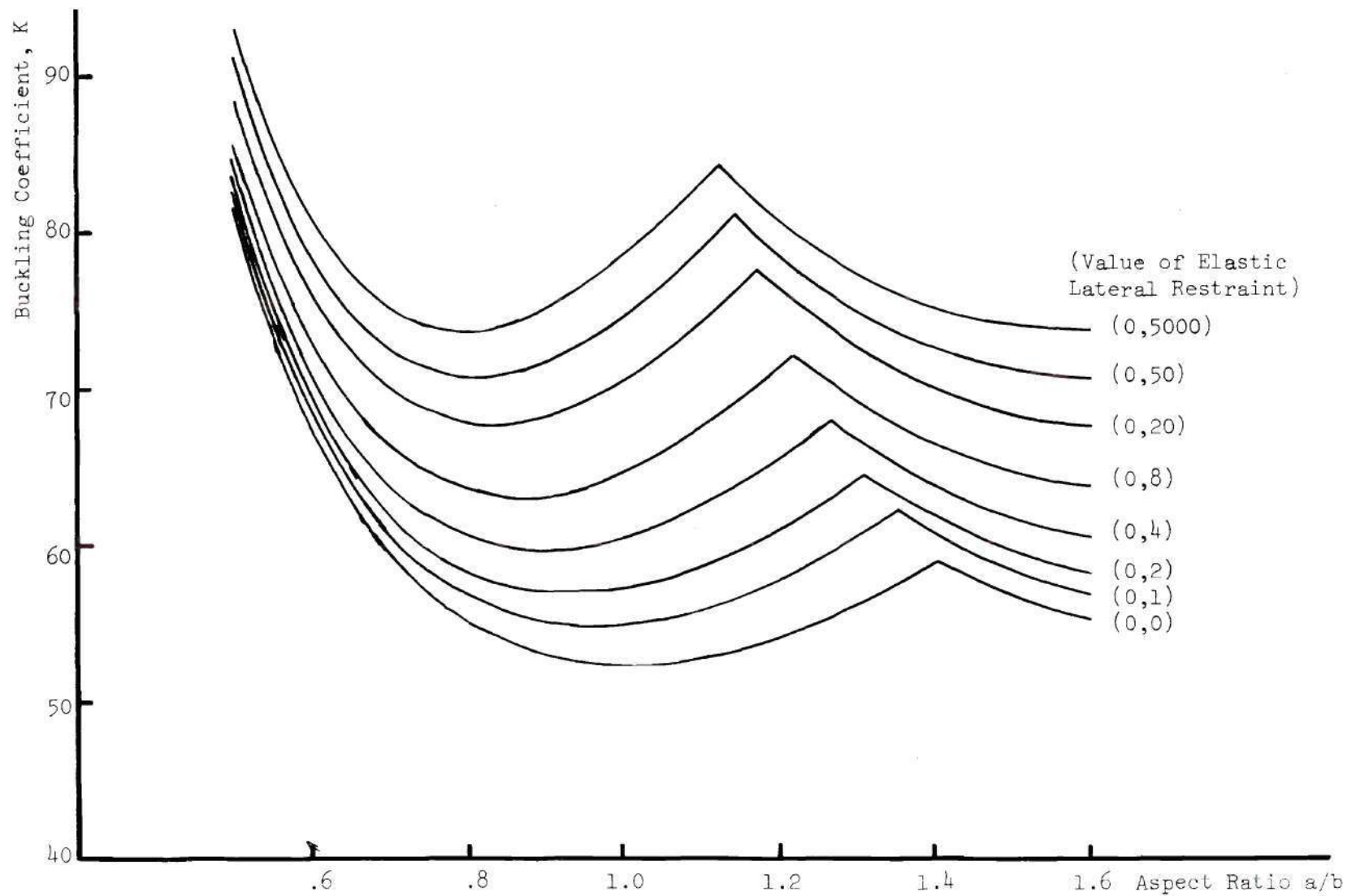


Figure 36. Buckling Coefficients Trapezoidal Load (0,β) Lateral Edges.

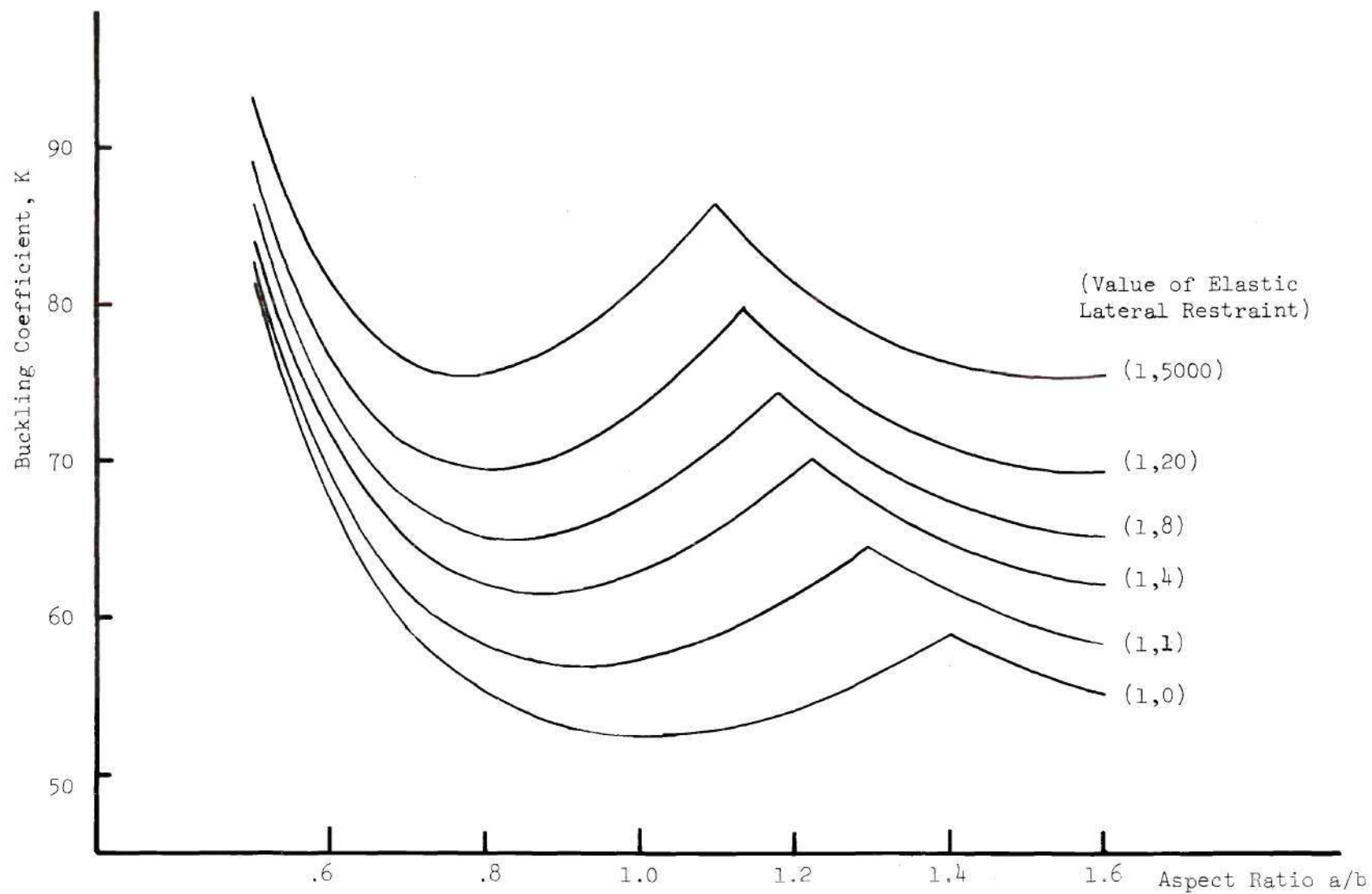


Figure 37. Buckling Coefficients Trapezoidal Load (1,3) Lateral Edges.

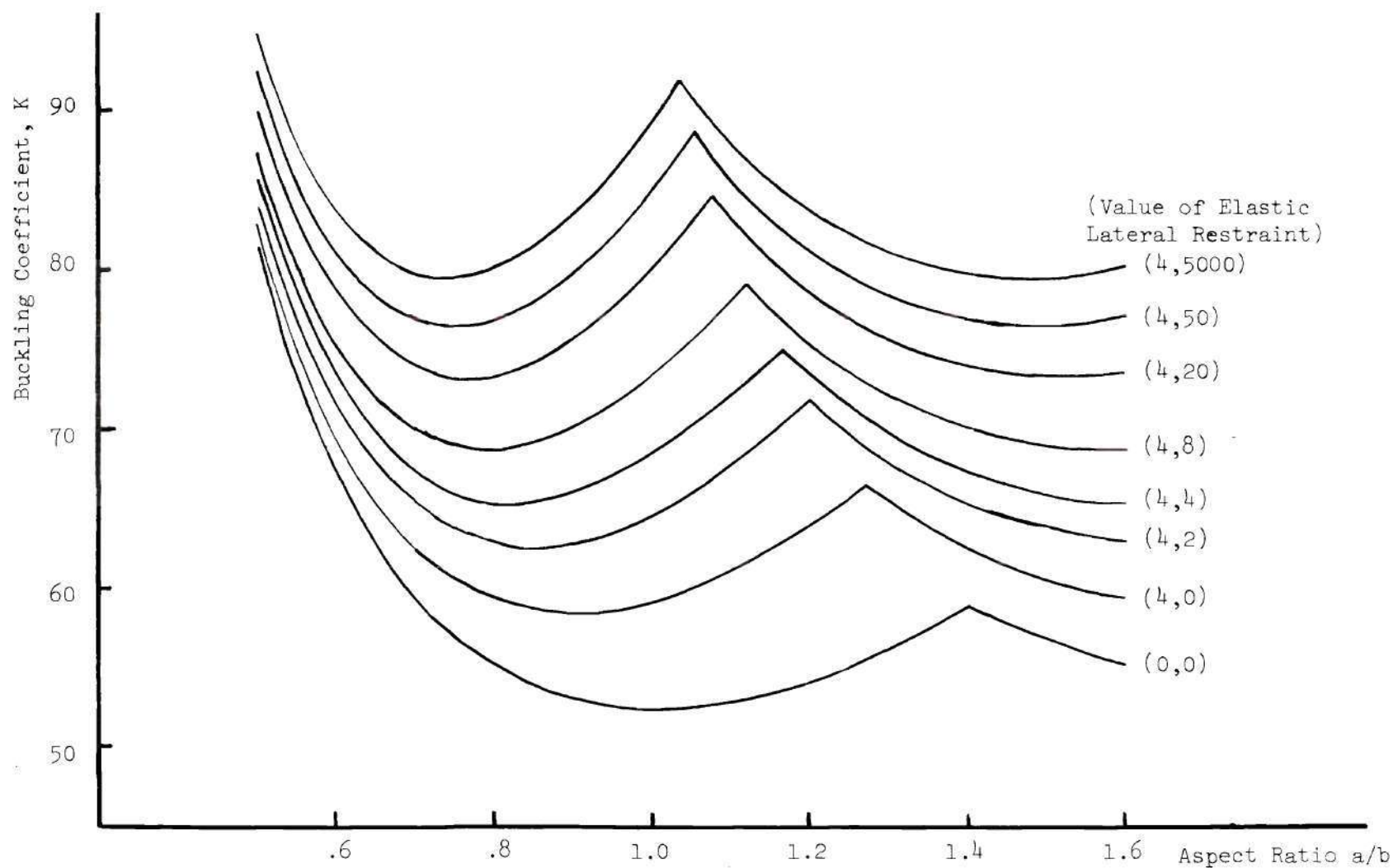


Figure 38. Buckling Coefficients Trapezoidal Load (4,  $\beta$ ) Lateral Edges.



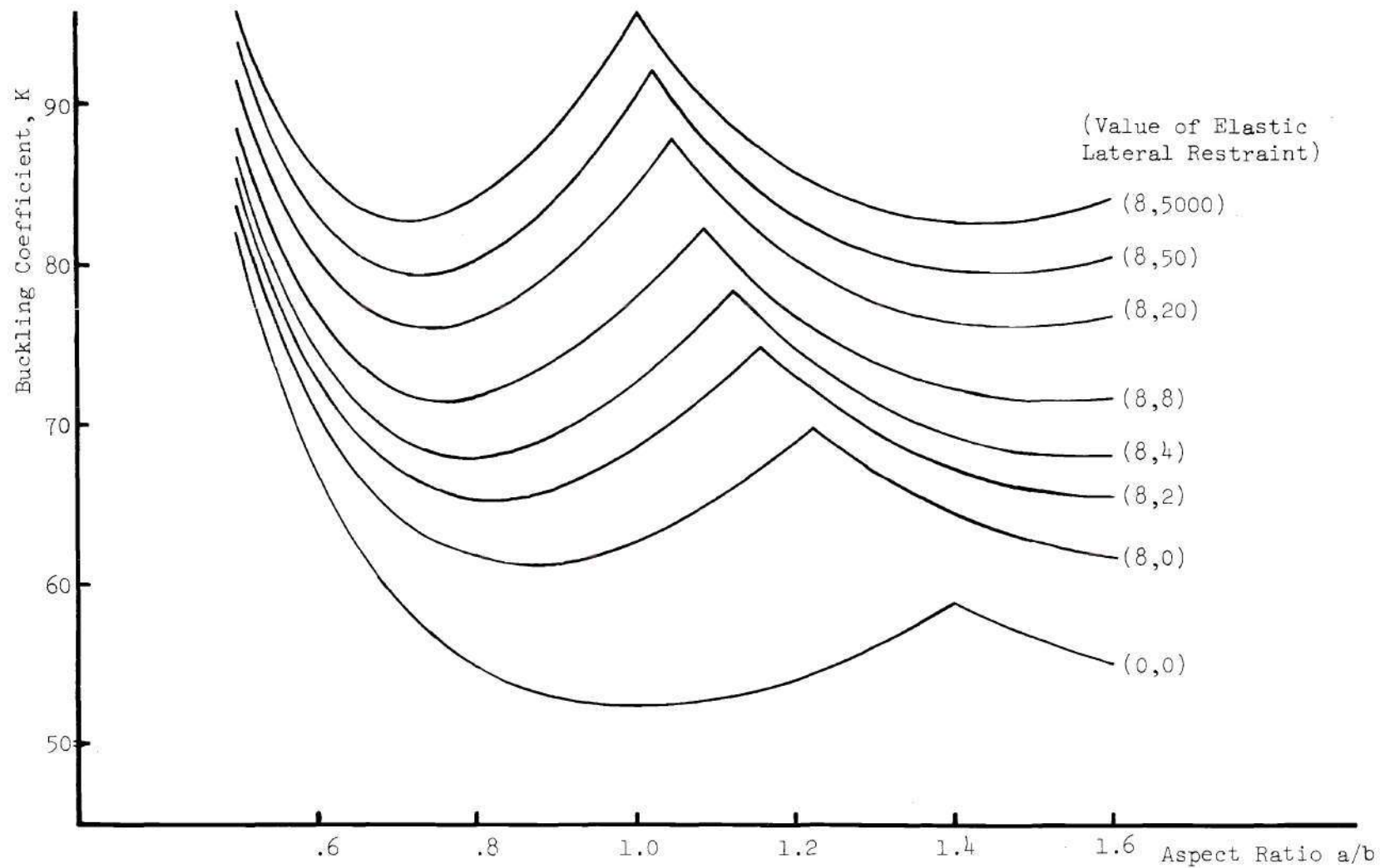


Figure 39. Buckling Coefficients Trapezoidal Load (8,8) Lateral Edges.

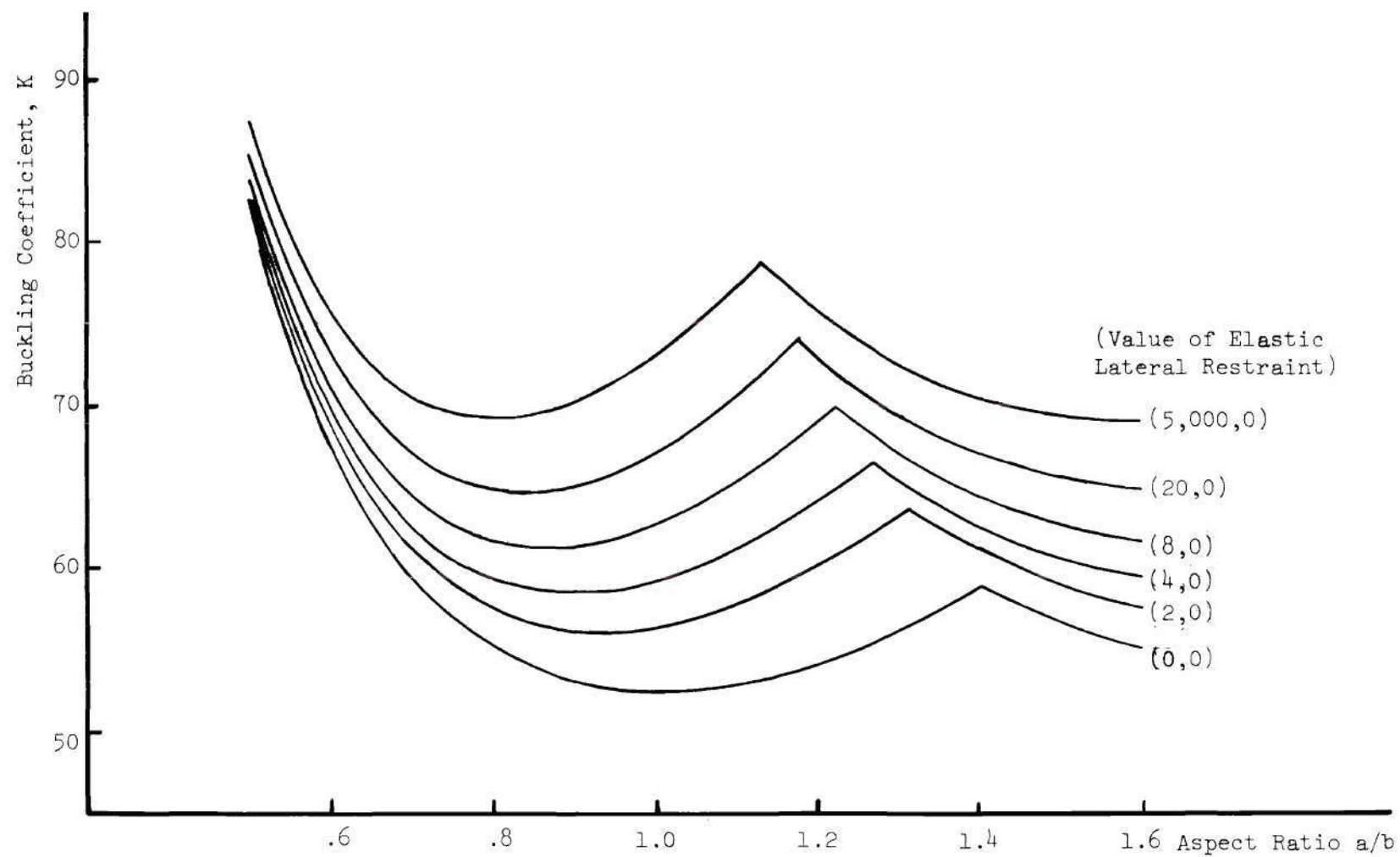


Figure 40. Buckling Coefficients Trapezoidal Load  $(\alpha, 0)$  Lateral Edges.

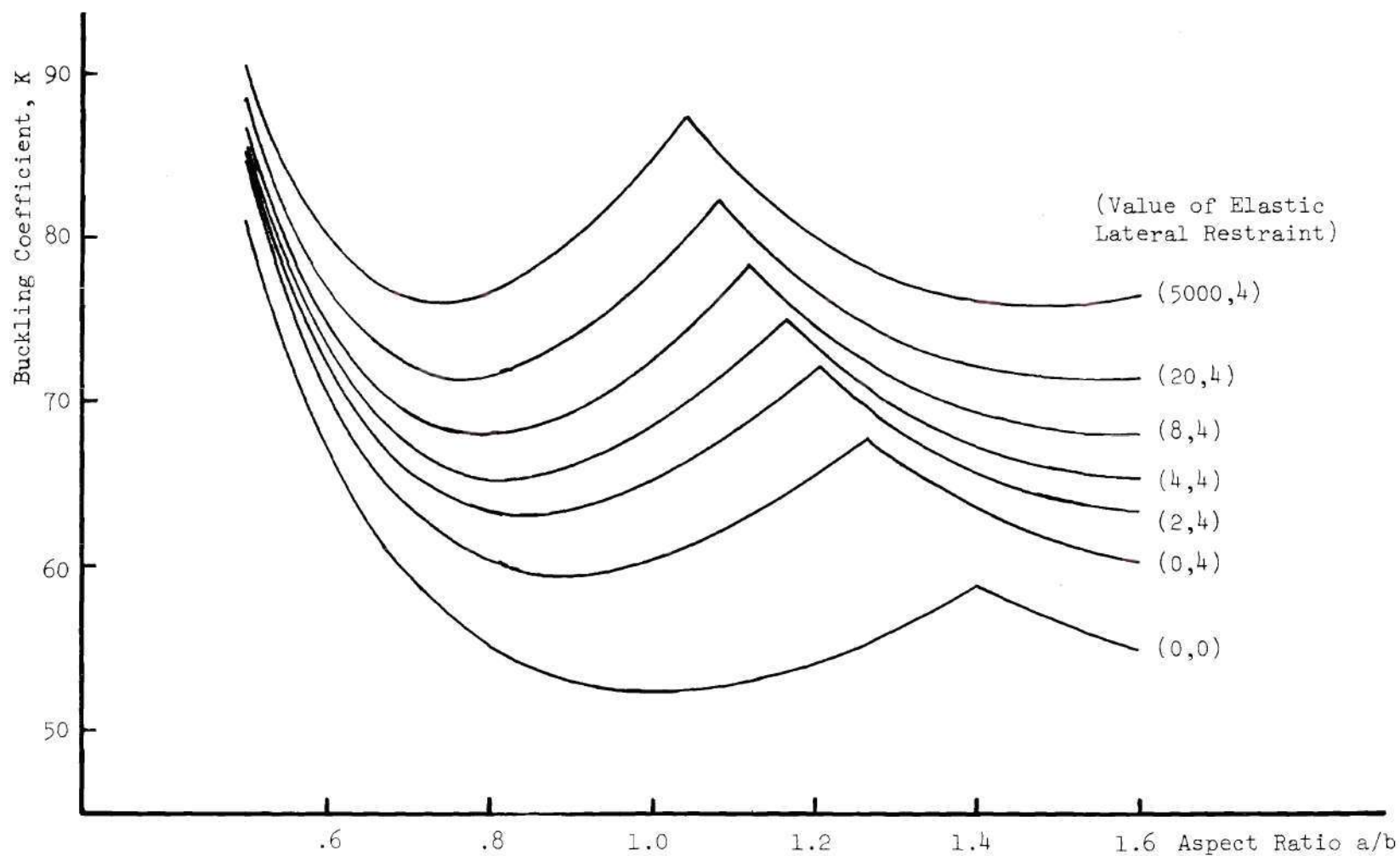


Figure 41. Buckling Coefficients Trapezoidal Load  $(\alpha, 4)$  Lateral Edges.

Thus, comparisons were readily made and excellent agreement was obtained, as shown in Table 2. The buckling coefficient curves for the various boundary restraint cases are shown in Figures 42 through 46. The difference in definition of  $K$  from common use accounts for the numerical shift.

In order to further verify that the distribution of load is influential only in the loading parameter function,  $\alpha_L$ , the computations were repeated for the uniform loading case using a displacement function more properly associatable with a trapezoidal loading action. This profile can be formed by the addition of a triangular load distribution to a uniform load distribution. It will be proportional to the arithmetic mean of the beam deflection functions for the beam with a triangular lateral load and the beam with a uniform lateral load. The procedure will be illustrated for the case of a simply supported plate. In this case, the appropriate beam deflection functions are

$$z_1 = (\lambda/3)(3y^5/\ell^2 - 10y^3 + 7\ell^2 y) \quad (35)$$

for the beam with a triangular lateral load and

$$z_2 = (5\lambda/2)(\ell^2 y - 2y^3 + y^4/\ell) \quad (36)$$

for the beam with an uniform lateral load. Forming the arithmetic mean,

$$z_{\text{trapezoidal}} = (z_1 + z_2)/2, \quad (37)$$

results in the following expression for the deflection function of beam with a trapezoidal lateral load and simply supported ends, viz.

Table 2. Comparison of the Buckling Coefficients Given by Lundquist and Stowell and Those of the Present Analysis.

$\beta, \beta$	Aspect Ratio											
	Ref. 145	Present Analysis	% Error	Ref. 145	Present Analysis	% Error	Ref. 145	Present Analysis	% Error	Ref. 145	Present Analysis	% Error
0,0	50.710	50.827	.23	41.482	41.641	.38	39.478	39.700	.56	40.801	41.098	.73
2,2	53.247	53.454	.39	45.894	46.210	.68	46.318	46.767	.96	50.592	51.216	1.22
6,6	56.809	57.097	.50	51.845	52.326	.92	55.319	56.062	1.32			
10,10	59.158	59.471	.53	55.625	56.195	1.01	60.955	61.852	1.45			
50,50	66.176	66.481	.46	66.492	67.160	.99	Lundquist and Stowell Only Tabulated Values of K for Buckling in One Half Wave					
$\infty, \infty$ (5000, 5000)	70.084	70.297	.30	72.305	72.870	.78						



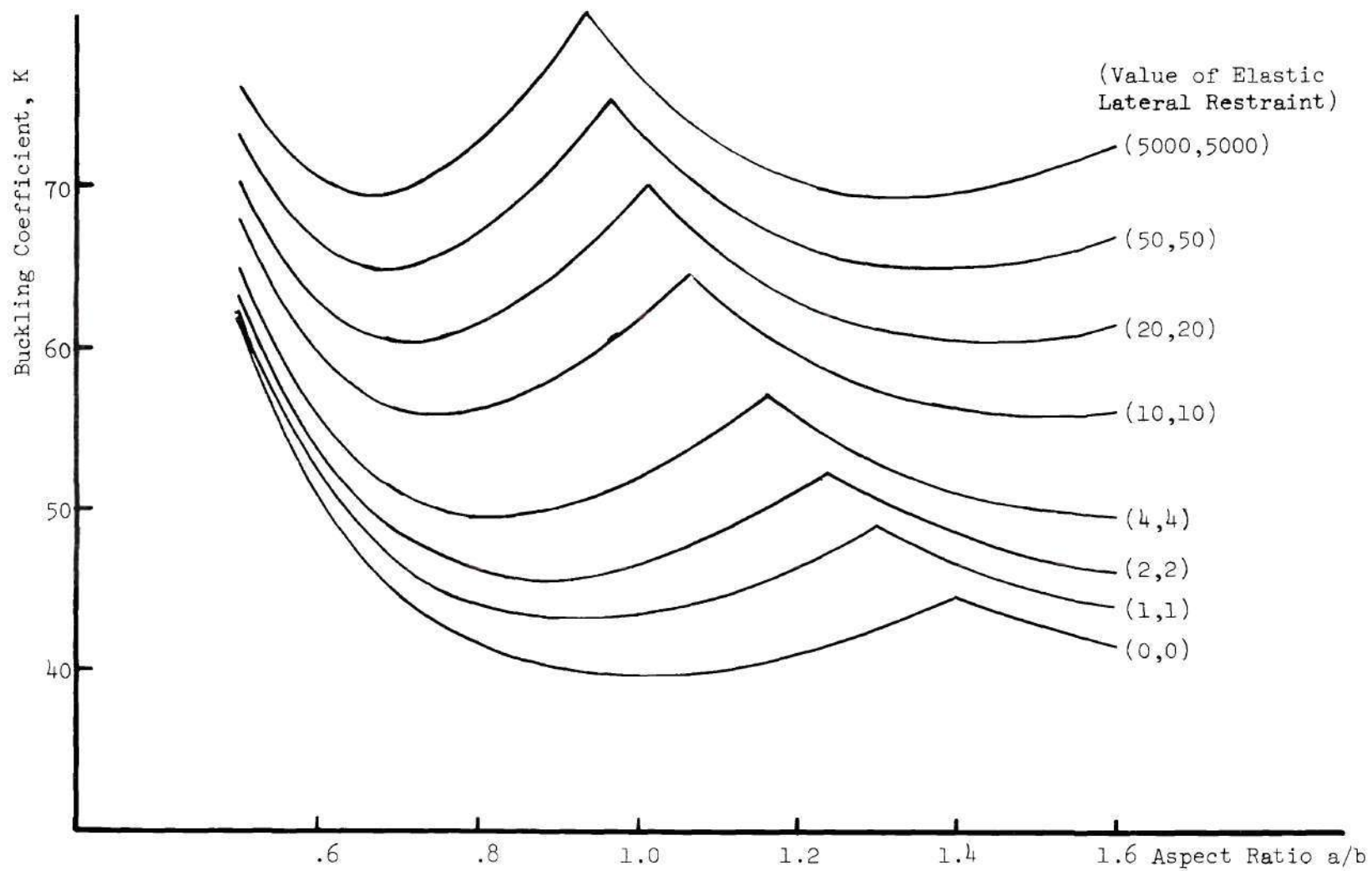


Figure 42. Buckling Coefficients Uniform Load  $(\beta, \beta)$  Lateral Edges.

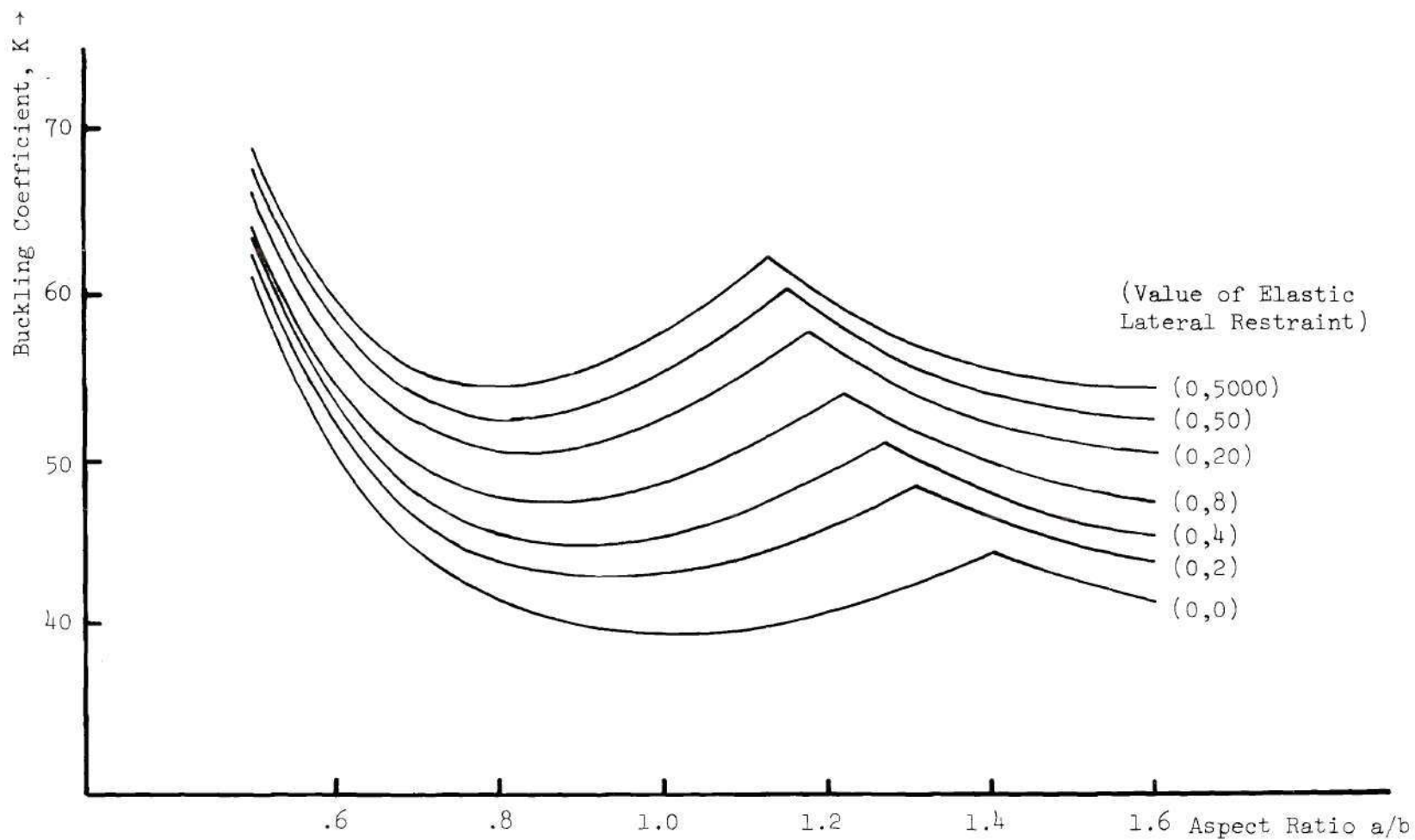


Figure 43. Buckling Coefficients Uniform Load (0,8) Lateral Edges.

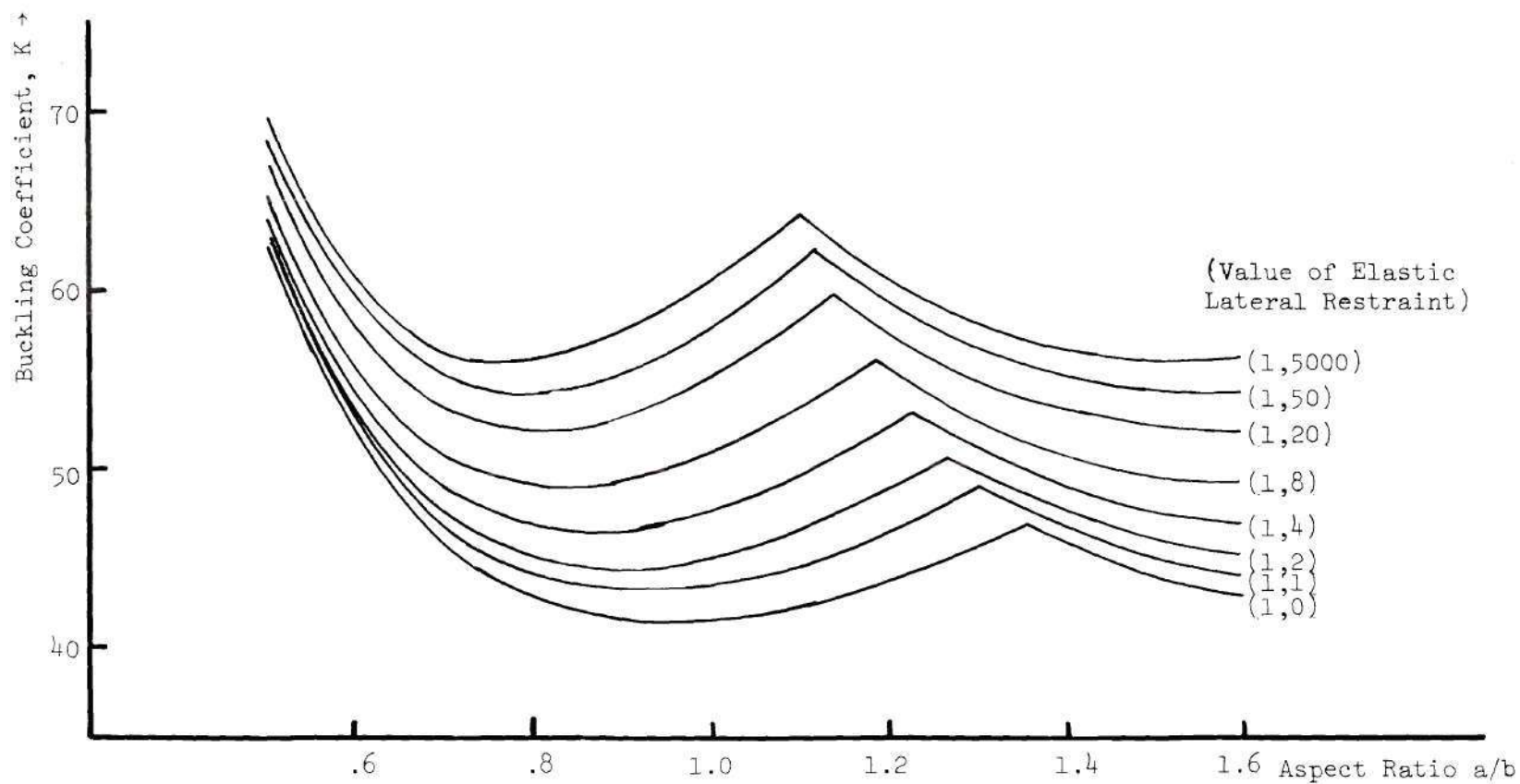


Figure 44. Buckling Coefficients Uniform Load (1,8) Lateral Edges.

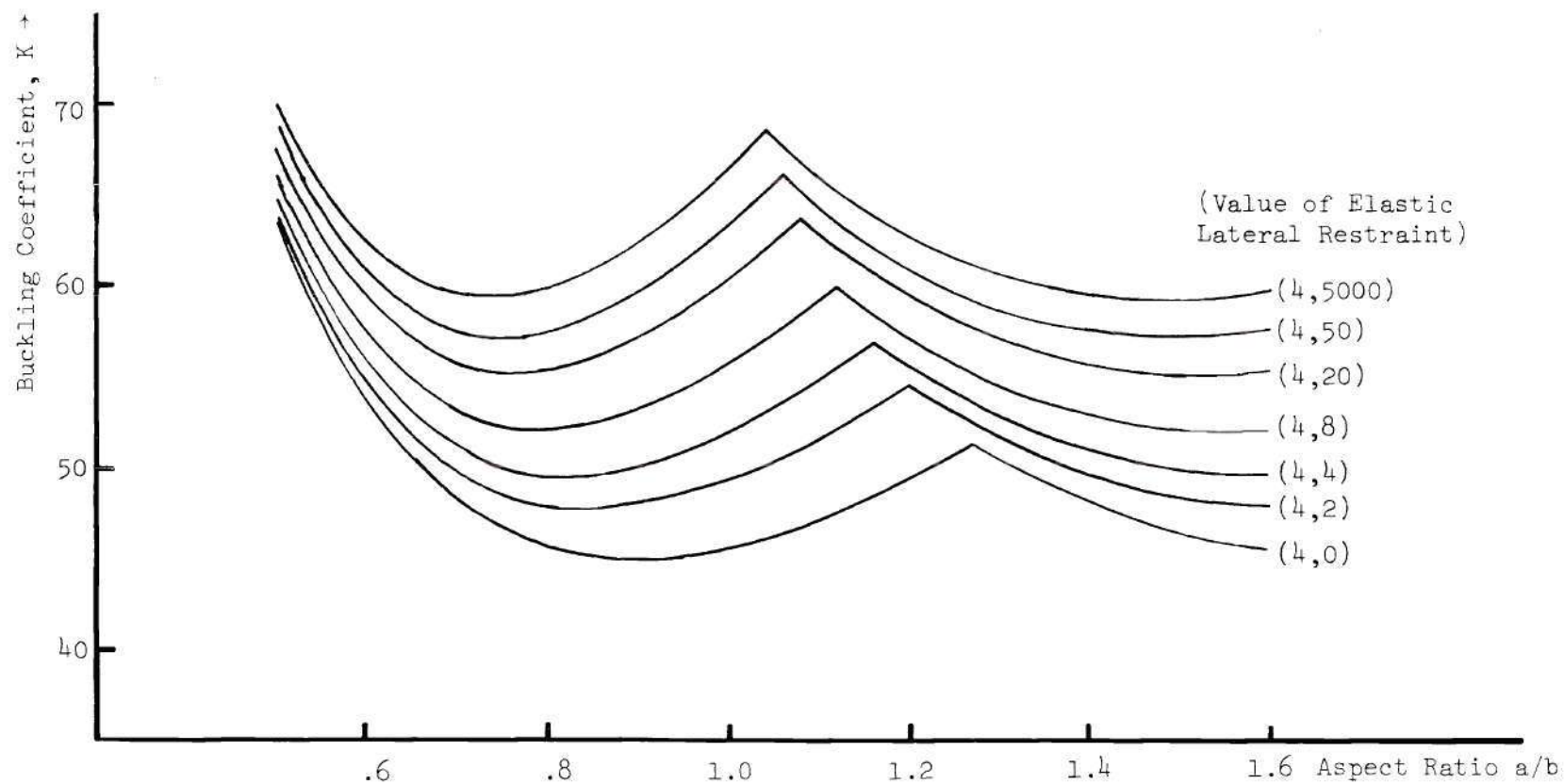


Figure 45. Buckling Coefficients Uniform Load (4, $\beta$ ) Lateral Edges.

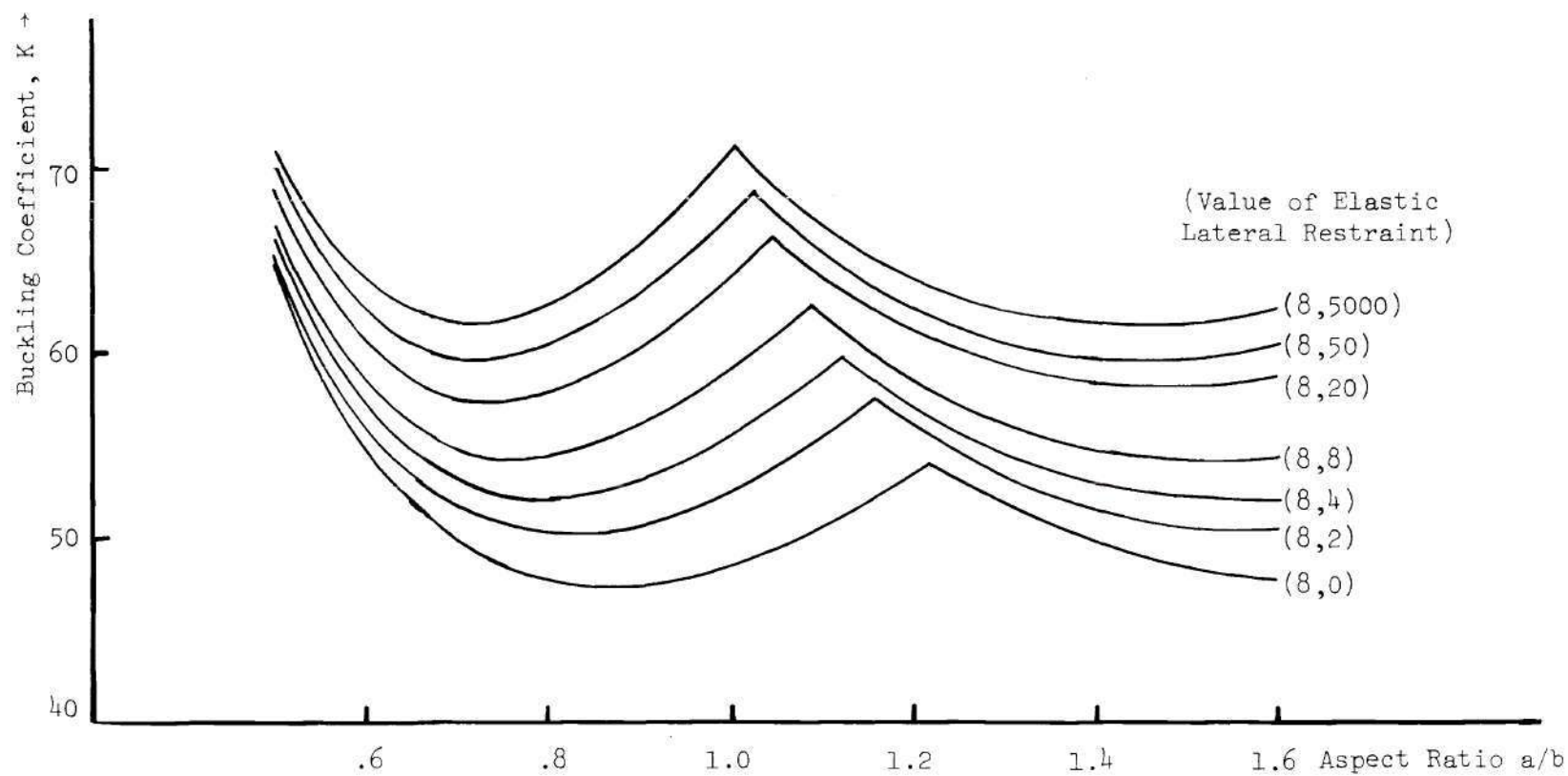


Figure 46. Buckling Coefficients Uniform Load  $(8, \beta)$  Lateral Edges.



$$z_{\text{trapezoidal}} = (\lambda/3)(6y^5/\ell^2 - 50y^3 + 29\ell^2y + 15y^4\ell) . \quad (38)$$

Hence, the deflection function for the plate may be written as

$$w(x,y) = (\lambda/3)(6y^5/b^2 - 50y^3 + 29b^2y + 15y^4b) \sin \frac{n\pi x}{a} \quad (39)$$

which is to be substituted into the Rayleigh Quotient as before. When this is done, with  $\alpha_L = 1$  for uniform load, the value of  $K$  which results is virtually identical (0.2 percent error) to the well known standard value obtained by using the classical deflection function, namely, a sine curve. The difference between using the deflection function for triangular or trapezoidal load and using the accepted deflection function for uniform load is so small, in the uniform load case, that the question naturally arises as to the possibility of using the deflection function for uniform load in the triangular load case. The deflection function for a beam with simply supported ends and a uniform lateral load distribution is given as

$$z = (5\lambda/2)(\ell^2y - 2y^3 + y^4/\ell) \quad (40)$$

so that the corresponding plate deflection function becomes

$$w(x,y) = (5\lambda/2)(b^2y - 2y^3 + y^4/b) \sin \frac{n\pi x}{a} \quad (41)$$

When this deflection function is utilized for the case of a triangular distribution of axial compression, the agreement is well within acceptable engineering accuracy (1.7% error). If the previously derived

trapezoidal deflection function is used in the triangular load case, the accuracy is even better (0.7% error).

In view of the good accuracy obtained by using several different deflection functions in each load case, one must reach the conclusion that the loading profile parameter,  $\alpha_L$ , is of greater influence in determining the value of the buckling coefficient  $K$  than is the choice of deflection curve. Of course, substantial excursions from the true shape would probably be prohibited but these results indicated that greater latitude in deflection shape selection is possible than has previously been thought provided the boundary conditions are satisfied.

## CHAPTER IV

### APPLICATIONS OF RATIONAL FUNCTIONS

#### Introduction

In Chapter III, buckling coefficients were obtained for plates, with various combinations of lateral boundary conditions, subject to triangular, trapezoidal or uniform distributions of axial compression. In the majority of cases, these were the first solutions for these configurations of load and boundary conditions. However, engineering research must also be directed along lines of consolidation and simplification. As recently stated by Fung and Sechler [148] "The ancients lost their records because they lacked paper to write on. The moderns, on the contrary, may lose theirs by having records drown in an ocean of printed matter. Therefore, there is a need to review, to consolidate, to simplify, to put the record in an accessible form." The practicing engineer as well as the designer must often resort to concise approximations in order to expedite a rapid solution. Even the electronic computer is not a panacea since it often requires a definite "turn around" time before a solution is obtained and any computer processing entails expense. Hence, there is justification for efforts to provide simple yet accurate approximations in engineering. Such research has been pursued recently by Struble [149], Iwamoto [150], Bank [151], Singhal [152], and Chaudhari [153] for the stability of columns and by Nassar [146] for the vibration of beams and plates.

In the present study, extensive use will be made of rational functions [154], [155]. Succinctly stated, a rational function is the ratio of two polynomials; in its simplest form, it is the ratio of two first-order polynomials. Such a function has one singular and distinct property which makes it eminently suited to studies on plate stability. It will generally retain a finite value even though the independent variable may range from zero to infinity. This property will be utilized to form approximate expressions for the buckling coefficients of plates with elastic rotational restraints along their lateral edges. Thereby, the restraint parameter may vary from zero (simple-support) to infinity (fixed or built-in) but, even so, the values of the buckling coefficients remain finite.

#### Effective Aspect Ratio

Before determining rational function approximations for the values of the buckling coefficients, attention must be paid to the question of aspect ratio. The buckling load of a uniaxially compressed plate is dependent upon the number of half waves in the longitudinal (loaded) direction, the aspect ratio of the plate and the boundary conditions. The influence of the number of half waves manifests itself in the familiar scalloped form of the curves of buckling coefficient versus aspect ratio. This scalloped form is readily apparent in the curves given in Chapter II for triangular and trapezoidal as well as uniform distributions. Examination of these curves reveals that the intersection points of consecutive wave number curves and the minimum points



corresponding to a given wave number lie on smooth curves whose equation can be expressed in a rational function form. The shift along this curve of the minimum buckling coefficients may be explained in the following manner; as the boundary restraint is increased, the locus of inflexion points moves inward from the edges of the plate. This can be seen in all of the previous figures by observing that the shift in half-wave intersection points is from right to left as one goes from simply-supported to fixed boundary restraint. That is to say that the fixed plate will transition from one to two half waves at a smaller aspect ratio than the simply supported plate.

A rational function approximation to the equation of the locus of transition points will now be established for one specific case. The case chosen is that of a plate which is subjected to a triangular distribution of compressive load. The pertinent boundary conditions are taken as, simply supported loaded edges with equal rotational elastic restraint along the unloaded lateral edges. This computation will serve to illustrate the procedure to be followed. In accordance with the procedures developed in prior researches (147) (149), and (150) the rational function will be taken in the form

$$AR_{\beta\beta} = \frac{\beta \cdot AR_{\infty\infty} + C \cdot AR_{00}}{\beta + C} \quad (42)$$

where AR represents aspect ratio and the suffixes imply edge condition. Thus  $\infty$  implies clamped, 0 implies simply supported, and  $\beta$  that the edge rotational restraint parameter has a value of  $\beta$ . C is a matching



coefficient. Examination of Figure 21 results in the following values of aspect ratio at the one to two half wave transition points, namely,  $AR_{\infty} = .934$  and  $AR_{00} = 1.4$ . As implied by the definition of  $C$  above, the rational function must be "matched at a third point" to the transition point wave. The point selected for this case will be the transition point on the  $\beta = 10$  curve. This curve, of course, lies intermediate between the  $\beta = 0$  and  $\beta = \infty$  curves. For the  $\beta = 10$  curve, the transition from one to two half waves occurs at an aspect ratio of 1.063. Thus,

$$AR_{\beta\beta} = \frac{\beta \cdot AR_{\infty} + C \cdot AR_{00}}{\beta + C} \quad (42)$$

now becomes

$$AR_{10,10} = 1.063 = \frac{10(.934) + C(1.4)}{10 + C} \quad (43)$$

which is to be solved to determine  $C$ . Hence,

$$10.63 + 1.063 C = 9.34 + 1.4C \quad (44)$$

or

$$.337 C = 1.29 \quad (45)$$

which yields

$$C = 3.828 \quad (46)$$

so the rational function is now expressed as

$$AR_{\beta\beta} = \frac{.934\beta + 5.3592}{\beta + 3.828} \quad (47)$$

which is the rational function approximation for the aspect ratio as a function of  $\beta$ . The overall accuracy of this approximation needs to be assessed. Hence, a comparison of the value given by the approximation with the actual value on the curves for  $\beta = 20$  and  $\beta = 2$  will now be made. For  $\beta = 20$ ,

$$\begin{aligned} AR_{20,20} &= \frac{20(.934) + 5.3592}{20 + 3.828} \\ &= \frac{24.03905}{23.828} = 1.0088 \end{aligned} \quad (48)$$

which is within 0.1 percent of the true value while for  $\beta = 2$

$$AR_{2,2} = \frac{2(.934) + 5.3592}{2 + 3.828} = 1.24 \quad (49)$$

which coincides precisely with the value from the curve. Because the locus of intersection points is a continuous non-oscillatory curve and the rational function is likewise, it follows that since they virtually coincide at five equally spaced points, each could be taken to represent the other with good accuracy. Using the procedure outlined above, similar rational function representations can be obtained for the other figures given in Chapter II. These have been tabulated into the following table.

All of these rational function representations yield values for the effective aspect ratio which are within 0.3 percent of the values obtained by directly reading the curves. By using these representations,

Table 3. Rational Function Representations for Effective Aspect Ratio Under Triangular Axial Load; as Determined from the One to Two Half Wave Transition Points, for Various Combinations of Elastic Lateral Restraint.

$\alpha, \beta$	Rational Function Representation
$\beta, \beta$	$AR_{\beta, \beta} = \frac{.934\beta + 5.3592}{\beta + 3.828}$
$0, \beta$	$AR_{0, \beta} = \frac{1.12\beta + 5.6602}{\beta + 4.043}$
$1, \beta$	$AR_{1, \beta} = \frac{1.09\beta + 5.47}{\beta + 4.046}$
$4, \beta$	$AR_{4, \beta} = \frac{1.037\beta + 5.4778}{\beta + 4.32}$
$8, \beta$	$AR_{8, \beta} = \frac{1.006\beta + 5.4778}{\beta + 4.4638}$
$\alpha, 0$	$AR_{\alpha, 0} = \frac{1.125\alpha + 5.723}{\alpha + 4.0879}$
$\alpha, 4$	$AR_{\alpha, 4} = \frac{1.04\alpha + 5.6879}{\alpha + 4.507}$

the effective aspect ratio can readily be calculated for a given set of lateral boundary conditions.

Having obtained rational functions when the load was of a triangular distribution, the next logical step is to repeat the procedure for the trapezoidal and uniform load distributions. Evaluation of the effective aspect ratio rational functions for the trapezoidal and the uniform cases confirms the suspicion aroused by a visual comparison of the curves. The rational functions for the trapezoidal and uniform cases are the same as those for the triangular distribution case. Thus, the effective aspect ratio is independent of loading distribution for these cases.

#### Buckling Coefficients

Having accounted for the aspect ratio change due to boundary restraint by the concept of an effective aspect ratio which can be expressed by rational functions, the rational function approach will be applied now to the buckling coefficients themselves. A similar form of the rational function approximation will be assumed for the buckling coefficients as was assumed for effective aspect ratio, viz.

$$K_{\beta\beta} = \frac{\beta K_{\infty\infty} + CK_{00}}{\beta + C} \quad (50)$$

where,  $K_{\infty\infty}$  is the buckling coefficient in the fixed ( $\beta = \infty$ ) case, at the approximate effective aspect ratio.

$K_{00}$  is the buckling coefficient in the simply-supported case ( $\beta = 0$ ), at the approximate effective aspect ratio

$K_{\beta\beta}$  is the buckling coefficient when the elastic rotational restraint parameter is equal to  $\beta$ , along the unloaded edge, at the appropriate effective aspect ratio

$\beta$  is the value of elastic rotational restraint parameter which varies from 0 to  $\infty$

C is a coefficient to be determined by matching at an intermediate value (third point) of aspect ratio and  $\beta$ .

For the reasons previously mentioned, only when the appropriate effective aspect ratio concept is used will comparisons between  $K_{\infty\infty}$ ,  $K_{00}$ ,  $K_{\beta\beta}$ , etc., be possible and valid. To determine the appropriate effective aspect ratio, the rational function representations developed earlier must be used. In the following derivation, the buckling coefficient values at the one to two half wave transition points will be selected since they clearly lie on the same effective aspect ratio curve. The case to be considered is that of equal elastic lateral restraints,  $\beta\beta$  case, where  $K_{\infty\infty} = 157.5$  and  $K_{00} = 87.2$ . In this instance, the intermediate or "third point" will be chosen as  $K_{66} = 117$ . Hence, the form

$$K_{\beta\beta} = \frac{\beta K_{\infty\infty} + C K_{00}}{\beta + C} \quad (50)$$

becomes

$$K_{66} = 117 = \frac{6(157.5) + 87.2C}{6 + C} \quad (51)$$

which must be solved for C. Therefore,



$$702 + 117C = 945 + 87.2C \quad (52)$$

and

$$29.8C = 243 \quad (53)$$

so that

$$C = 8.1544 \quad (54)$$

Thus, the rational function becomes

$$K_{\beta\beta} = \frac{\beta K_{\infty\infty} + 8.1544 K_{00}}{\beta + 8.1544} = \frac{157.5\beta + 8.1544(87.2)}{\beta + 8.1544} \quad (55)$$

or

$$K_{\beta\beta} = \frac{157.5\beta + 711.0604}{\beta + 8.1544} \quad (56)$$

Evaluating this rational function approximation at  $\beta = 10$  yields 125.9232 which agrees well (within 0.4 percent) with the value of 125.3 from the curve. At  $\beta = 2$ , the rational function yields 101.0459 while the proper value obtained from the curve is 101.8 which is an error of only 0.75 percent. As was done for aspect ratios, rational functions representations for buckling coefficients were obtained for the various boundary condition cases. These are given in Table 4.

Similar rational functions may be determined for the buckling coefficients for plates subject to a trapezoidal distribution of axial compression as well as uniform compression. These are given in Tables 5 and 6, respectively.

Table 4. Rational Function Representation for Buckling Coefficients Under Triangular Axial Load; as Determined From the One-to-Two Half Wave Transition Points, for Various Combinations of Elastic Lateral Restraint.

$\alpha, \beta$	Rational Function Representation
$\beta, \beta$	$K_{\beta, \beta} = \frac{157.5\beta + 711.0604}{\beta + 8.1544}$
$0, \beta$	$K_{0, \beta} = \frac{135.6\beta + 701.2979}{\beta + 8.0332}$
$1, \beta$	$K_{1, \beta} = \frac{138.1\beta + 755.5214}{\beta + 8.3761}$
$4, \beta$	$K_{4, \beta} = \frac{143.25\beta + 848.0441}{\beta + 8.8987}$
$8, \beta$	$K_{8, \beta} = \frac{146.8\beta + 917.6432}{\beta + 9.273}$
$\alpha, 0$	$K_{\alpha, 0} = \frac{107.7\alpha + 556.856}{\alpha + 6.386}$
$\alpha, 4$	$K_{\alpha, 4} = \frac{122.3\alpha + 743.76}{\alpha + 7.2}$

Table 5. Rational Function Representation for Buckling Coefficients Under Trapezoidal Axial Load; as Determined from the One-to-Two Half Wave Transition Points, for Various Combinations of Elastic Lateral Restraint.

$\alpha, \beta$	Rational Function Representation
$\beta, \beta$	$K_{\beta, \beta} = \frac{107.5\beta + 479.375}{\beta + 8.125}$
$0, \beta$	$K_{0, \beta} = \frac{84.5\beta + 432.647}{\beta + 7.333}$
$1, \beta$	$K_{1, \beta} = \frac{86.5\beta + 468.444}{\beta + 7.555}$
$4, \beta$	$K_{4, \beta} = \frac{92\beta + 532}{\beta + 8}$
$8, \beta$	$K_{8, \beta} = \frac{95.8\beta + 569.884}{\beta + 8.1412}$
$\alpha, 0$	$K_{\alpha, 0} = \frac{78.8\alpha + 387.04}{\alpha + 6.56}$
$\alpha, 4$	$K_{\alpha, 4} = \frac{87.6\alpha + 489.6}{\alpha + 7.2}$

Table 6. Rational Function Representation for Buckling Coefficients Under Uniform Axial Load; as Determined from the One-to-Two Half Wave Transition Points, for Various Combinations of Elastic Lateral Restraint.

$\alpha, \beta$	Rational Function Representation
$\beta, \beta$	$K_{\beta, \beta} = \frac{80.8\beta + 338.2464}{\beta + 7.584}$
$0, \beta$	$K_{0, \beta} = \frac{62.3\beta + 307.9746}{\beta + 6.90526}$
$1, \beta$	$K_{1, \beta} = \frac{64.4\beta + 336.4208}{\beta + 7.1579}$
$4, \beta$	$K_{4, \beta} = \frac{68.5\beta + 400.9654}{\beta + 7.8161}$
$8, \beta$	$K_{8, \beta} = \frac{71.3\beta + 427.032}{\beta + 7.9080}$
$\alpha, 0 = 0, \beta$	By Symmetry
$\alpha, 4 = 4, \beta$	

### Buckling Coefficient Relationships

Now that values of the buckling coefficient  $K$  have been determined for many combinations of lateral boundary restraint and load distributions, the question of interest is whether or not there are relationships among these various  $K$ 's. If such relationships can be discovered then all round simplicity will result.

In this study, the first relationship discovered is given as:

$$K_{\infty\infty} + K_{00} = K_{0\infty} + K_{\infty 0} \quad (57)$$

The rationale which was utilized to reach this formulation may be simply stated. It was conjectured that a relationship should exist among the four extreme cases of the buckling coefficients which are  $K_{\infty\infty}$ ,  $K_{00}$ ,  $K_{0\infty}$ , and  $K_{\infty 0}$ . Since  $K_{0\infty}$  and  $K_{\infty 0}$  are intermediate in value between  $K_{\infty\infty}$  and  $K_{00}$ , the possibility of a mean value law immediately suggests itself. Due to its simplicity such a law should be one of the first to be investigated. Hence, the question becomes,

$$\frac{K_{\infty\infty} + K_{00}}{2} \stackrel{?}{=} K_{0\infty} \quad (58)$$

to which the answer is no. However, the relationship

$$\frac{K_{\infty\infty} + K_{00}}{2} = K_{0\infty} + C_1 \quad (59)$$

is correct. The next relationship to be examined is whether

$$\frac{K_{\infty\infty} + K_{00}}{2} \stackrel{?}{=} K_{\infty 0} \quad (60)$$



and again the answer is no as one might expect from the prior result for  $K_{0\infty}$ . But the relation, motivated by the  $K_{0\infty}$  case, i.e.,

$$\frac{K_{\infty\infty} + K_{00}}{2} = K_{\infty 0} - C_2 \quad (61)$$

is also correct. Fortunately

$$C_1 \approx C_2 \quad (62)$$

and the two relations

$$\frac{K_{\infty\infty} + K_{00}}{2} = K_{0\infty} + C_1 \quad (59)$$

$$\frac{K_{\infty\infty} + K_{00}}{2} = K_{\infty 0} - C_2 \quad (61)$$

may be added together to yield the relation that

$$K_{\infty\infty} + K_{00} = K_{0\infty} + K_{\infty 0} \quad (57)$$

which is accurate to within 0.6 percent in the case of a triangular or a trapezoidal distribution of load. Of course, in the uniform load case, from symmetry, there is the following equality

$$K_{0\infty} = K_{\infty 0} \quad (63)$$

so that the relationship becomes

$$\frac{K_{\infty\infty} + K_{00}}{2} = K_{0\infty} = K_{\infty 0} \quad (64)$$

which is a mean law relationship, as suggested by Lundquist and Stovall [140].

This first relationship can be extended to increase its generality. It is, obviously, a formula which allows one of K's to be determined in terms of the other three. Hence, this expression may be rewritten as

$$K_{\infty\infty} = K_{0\infty} + K_{\infty 0} - K_{00} \quad (65)$$

Now, this expression will be extended in the following manner. The value,  $K_{\infty\infty}$ , may be thought of as the extreme value of  $K_{\alpha\beta}$ . In keeping with this approach,  $K_{0\infty}$  and  $K_{\infty 0}$  will be regarded as the extreme values of  $K_{0\beta}$  and  $K_{\alpha 0}$ , respectively. Thus, the new form of the relationship is expressed as

$$K_{\alpha\beta} = K_{\alpha 0} + K_{0\beta} - K_{00} \quad (66)$$

The above relationship now permits consideration of a vastly greater number of lateral boundary condition cases. It serves to exemplify the common occurrence that a special case is found before the general one. It can be seen that this general expression satisfies the special case where  $\alpha = \beta = 0$  since it yields  $K_{00} = K_{00}$ . So the expression satisfies the special cases at its extremes which is the first stage in ascertaining its accuracy. When this expression is checked at various values of  $\alpha$  and  $\beta$ , it is found to be quite accurate as long as values which lie along the same effective aspect ratio curves are compared. This is best

demonstrated in the following table where the values of  $K_{\alpha 0}$ ,  $K_{0\beta}$  and  $K_{00}$  are taken from the curves given in Chapter II. As in the prior examples, values along the line of intersection of the one and two half-wave curves will be used, since this facilitates accurate determination of the values from the curves. Table 7 is a listing of results for a triangular load distribution.

The results presented in Table 7 demonstrate that this formulation is well within the bounds of engineering accuracy. Hence, it may be used as the basis for a further refinement. Versatile as this present relationship is nonetheless it does require that in order to determine  $K_{\alpha\beta}$  values of  $K_{\alpha 0}$  and  $K_{0\beta}$  be obtained each time that there is a change in either  $\alpha$  or  $\beta$ . It would be much more convenient if an expression could be formulated in terms of the extreme values of  $K$ , namely  $K_{\infty\infty}$ ,  $K_{0\infty}$ ,  $K_{\infty 0}$ , and  $K_{00}$  and the values of  $\alpha$  and  $\beta$ . Such a relationship would dispense with the requirement for determining a multiplicity of  $K_{\alpha 0}$ 's and  $K_{0\beta}$ 's. The problem would then be reduced to supplying only the six quantities given above where  $\alpha$  and  $\beta$  would vary but  $K_{\infty\infty}$ ,  $K_{0\infty}$ ,  $K_{\infty 0}$  and  $K_{00}$  would not.

Starting with the basis relationship,

$$K_{\alpha\beta} = K_{\alpha 0} + K_{0\beta} - K_{00} \quad (66)$$

It will be remembered that rational function representations have been derived for each of the inputs, i.e.,  $K_{\alpha 0}$ ,  $K_{0\beta}$  and  $K_{00}$  for triangular, and trapezoidal distributions of loading. For the triangular distribution, these are given in Table 4 as

Table 7. Comparison of Predicted and Actual  
Values of  $K_{\alpha\beta}$ .

$\alpha, \beta$	Input Values From Curves			Predicted	Actual	Percentage
	$K_{\alpha 0}$	$K_{0\beta}$	$K_{00}$	Value $K_{\alpha\beta}$	Value $K_{\alpha\beta}$	Error %
4,4	95.3	103.3	87.2	111.4	110.9	.45
1,8	90.2	111.3	87.2	114.2	113.6	.6
8,4	98.7	103.3	87.2	114.8	113.3	1.3
8,50	98.7	128.8	87.2	140.3	139.2	.8
20,4	102.8	103.3	87.2	118.9	117.2	1.4
4,50	95.3	128.8	87.2	136.9	136.0	.7
1,2	90.1	97.1	87.2	100.0	99.4	.6
10,10	99.5	114.1	87.2	126.4	125.3	.9
1,1	90.1	92.9	87.2	95.8	95.6	.2
50,4	105.2	103.3	87.2	121.3	120.0	1.1

$$K_{\alpha,0} = \frac{\alpha K_{\infty 0} + 6.386 K_{00}}{\alpha + 6.386} \quad (67)$$

$$K_{0,\beta} = \frac{\beta K_{0\infty} + 8.0332 K_{00}}{\beta + 8.0332} \quad (68)$$

where  $K_{\alpha,0}$  and  $K_{0,\beta}$  are expressed in terms of  $K_{\infty 0}$ ,  $K_{00}$  and  $K_{0\infty}$ ,  $K_{00}$ , respectively. Thus,

$$K_{\alpha\beta} = K_{\alpha,0} + K_{0,\beta} - K_{00} \quad (66)$$

may be written, using these rational function representations, as

$$\begin{aligned} K_{\alpha\beta} &= \frac{\alpha K_{\infty 0} + 6.386 K_{00}}{\alpha + 6.386} + \frac{\beta K_{0\infty} + 8.0332 K_{00}}{\beta + 8.0332} - K_{00} \quad (69) \\ &= [(\alpha K_{\infty 0} + 6.386 K_{00})(\beta + 8.0332) \\ &\quad + (\beta K_{0\infty} + 8.0332 K_{00})(\alpha + 6.386) \\ &\quad - (\alpha + 6.386)(\beta + 8.0332) K_{00}] / (\alpha + 6.386)(\beta + 8.0332) \\ &= \{ \alpha\beta(K_{\infty 0} + K_{0\infty} - K_{00}) + 8.0332\alpha K_{\infty 0} + 6.386\beta K_{0\infty} \\ &\quad + (8.0332)(6.386)K_{00} \} / [\alpha\beta + 6.386\beta + 8.0332\alpha \\ &\quad + (6.386)(8.0332)] \end{aligned}$$

Now, from the first relationship obtained, viz.

$$K_{\infty\infty} + K_{00} = K_{\infty 0} + K_{0\infty} \quad (57)$$

the expression  $(K_{\infty 0} + K_{0\infty} - K_{00})$  is equal to  $K_{\infty\infty}$  hence, the above



formulation becomes

$$K_{\alpha\beta} = \frac{\alpha\beta K_{\infty\infty} + 8.0332\alpha K_{\infty 0} + 6.386\beta K_{0\infty} + (8.0332)(6.386)K_{00}}{\alpha\beta + 6.386\beta + 8.0332\alpha + (6.386)(8.0332)} \quad (70)$$

Thus, it has been possible to derive a relatively simple expression for  $K_{\alpha\beta}$  in terms of  $K_{\infty\infty}$ ,  $K_{\infty 0}$ ,  $K_{0\infty}$ ,  $K_{00}$ ,  $\alpha$  and  $\beta$ . For this expression the values of  $K_{\infty\infty}$ ,  $K_{\infty 0}$ ,  $K_{0\infty}$  and  $K_{00}$  need to be determined only once while  $\alpha$  and  $\beta$  are varied to yield the desired  $K_{\alpha\beta}$ . As was the case in the relationship from which this present one was derived, care must be taken to use values from the correct effective aspect ratio curves. When this is done, the accuracy of the expression is good. One such set of admissible K's, those which result from considering effective aspect ratio, is given by

$$K_{00} = 78.3, K_{\infty 0} = 95, K_{\infty\infty} = 151 \quad \text{and} \quad K_{0\infty} = 119.2 \quad (71)$$

Employing these values in the  $K_{\alpha\beta}$  expression, where  $\alpha = 4$  and  $\beta = 2$ , yields

$$K_{\alpha\beta} = K_{4,2} = \frac{9799.83}{104.21} = 94.04 \quad (72)$$

while the correct value from the curves, again considering effective aspect ratio, is 93.4 which is an error of only 0.7 percent. When  $K_{8,50}$  is investigated using this formulation, the appropriate input values are

$$K_{00} = 77.9, K_{\infty\infty} = 148, K_{\infty 0} = 94.7, K_{0\infty} = 118.6 \quad (73)$$

which upon substitution into this expression for  $K_{\alpha\beta}$  yields

$$K_{8,50} = 128.35 \quad (74)$$

whereas the correct value as obtained from the curve is

$$K_{8,50} = 127.7 \quad (75)$$

which is an error of only 0.5 percent. This high degree of accuracy should not be surprising since the components from which the expression is constructed have been demonstrated to be quite accurate. The expression was derived for the triangular load distribution case, however, similar formulations will result for the trapezoidal and uniform load cases. The procedure to obtain these additional expressions is identical to the one given for the triangular load case.

For the trapezoidal load case:

$$K_{\alpha\beta} = \frac{\alpha\beta K_{\infty\infty} + 7.333\alpha K_{\infty 0} + 6.56\beta K_{0\infty} + (7.333)(6.56)K_{00}}{\alpha\beta + 7.333\alpha + 6.56\beta + (7.333)(6.56)} \quad (76)$$

For the uniform load case:

$$K_{\alpha\beta} = \frac{\alpha\beta K_{\infty\infty} + 6.9053(\alpha + \beta)K_{\infty 0} + (6.9053)^2 K_{00}}{\alpha\beta + 6.9053(\alpha + \beta) + (6.9053)^2} \quad (77)$$

There can be little doubt that the above expressions are of significant value in their present form. However, it would be remiss not to investigate the possibility of further simplification. There is a natural tendency to feel, due to the similar shape of each curve in the "family of curves," that only one curve is really necessary. It would appear logical to hypothesize the existence of a "primary curve"

from which the other curves could be generated by suitable operations. In order for this to occur, each curve in the "family" must be proportional to this "primary" curve. From the results presented thus far, a prime candidate for such a "primary" curve is that which corresponds to conditions, viz., the simply supported, the  $K_{00}$ , curve. The term,  $K_{00}$ , has occurred in each of the relationships for buckling coefficients and it represents the bottom curve in any given "family" of buckling coefficients. Furthermore,  $K_{00}$  has the largest value of effective aspect ratio since, it will be remembered, effective aspect ratio reflects the fact that the contour of inflexion points moves inward. Hence, the simply supported plate has the largest value of effective aspect ratio possible which is, of course, the actual aspect ratio.

The proportionality between  $K_{00}$  and  $K_{\infty\infty}$ ,  $K_{0\infty}$  and  $K_{\infty 0}$  may be expressed in the following manner:

$$K_{\infty\infty} = A_1 K_{00}, K_{0\infty} = B_1 K_{00}, K_{\infty 0} = C_1 K_{00} \quad (78)$$

where  $A_1$ ,  $B_1$ ,  $C_1$  are coefficients or factors of proportionality which must be determined from the buckling coefficient curves.

To utilize these expressions for  $K_{\infty\infty}$ ,  $K_{0\infty}$  and  $K_{\infty 0}$ , the previous relationship (for the triangular case)

$$K_{\alpha\beta} = \frac{\alpha\beta K_{\infty\infty} + 8.0332\alpha K_{\infty 0} + 6.386\beta K_{0\infty} + (8.0332)(6.386)K_{00}}{\alpha\beta + 6.386\beta + 8.0332\alpha + (6.386)(8.0332)} \quad (79)$$

will first be rewritten as

$$K_{\alpha\beta} = \frac{[\alpha\beta K_{\infty\infty} + C^* \alpha K_{\infty 0} + C\beta K_{0\infty} + C^* C K_{00}]}{\alpha\beta + C^* \alpha + C\beta + C^* C} \quad (80)$$

where  $C^* = 8.0332$  for triangular load

= 7.333 for trapezoidal load

= 6.9053 for uniform load

and

$C = 6.386$  for triangular load

= 6.56 for trapezoidal load

= 6.9053 for uniform load

Now the expressions  $K_{\infty\infty} = A_1 K_{00}$ ,  $K_{0\infty} = B_1 K_{00}$ , and  $K_{\infty 0} = C_1 K_{00}$  will be employed. First, these will be used in the relationship

$$K_{\infty\infty} + K_{00} = K_{0\infty} + K_{\infty 0} \quad (57)$$

which becomes

$$A_1 K_{00} + K_{00} = B_1 K_{00} + C_1 K_{00} \quad (81)$$

thus

$$A_1 K_{00} = (B_1 + C_1 - 1) K_{00} \quad (82)$$

or

$$A_1 = (B_1 + C_1 - 1) \quad (83)$$

Thus the relationship for  $K_{\alpha\beta}$  may be written as

$$K_{\alpha\beta} = \frac{[\alpha\beta(B_1 + C_1 - 1) + C^*\alpha B_1 + C\beta C_1 + C^*C] K_{00}}{\alpha\beta + C^*\alpha + C\beta + C^*C} \quad (84)$$

so that  $K_{\alpha\beta}$  is expressed in terms of only one buckling coefficient, namely,  $K_{00}$ .

For the case of a triangular load distribution, the following factors of proportionality are obtained directly from the buckling coefficient curves along the loci of transition from one to two half waves.

$$A_1 = \frac{K_{\infty\infty}}{K_{00}} = 1.8, B_1 = \frac{K_{0\infty}}{K_{00}} = 1.561, C_1 = \frac{K_{\infty 0}}{K_{00}} = 1.236 \quad (85)$$

and

$$B_1 + C_1 = 1.561 + 1.236 = 2.797 \approx 2.8 \quad (86)$$

so that, for a triangular load distribution,  $K_{\alpha\beta}$  is given by

$$K_{\alpha\beta} = \frac{[1.8\alpha\beta + 1.561(8.0332)\alpha + 1.236(6.386)\beta + (8.0332)(6.386)]K_{00}}{\alpha\beta + 8.0332\alpha + 6.386\beta + (8.0332)(6.386)} \quad (87)$$

which depends only upon  $\alpha$ ,  $\beta$  and  $K_{00}$ . For the trapezoidal load distribution, the buckling coefficient curves now directly yield the following factors of proportionality,

$$A_1 = \frac{K_{\infty\infty}}{K_{00}} = 1.8, B_1 = \frac{K_{0\infty}}{K_{00}} = 1.4322, C_1 = \frac{K_{\infty 0}}{K_{00}} = 1.3356 \quad (88)$$

and

$$B_1 + C_1 = 1.4332 + 1.3356 = 2.7678 \approx 2.8 \quad (89)$$

and, therefore, the appropriate expression for  $K_{\alpha\beta}$  is



$$K_{\alpha\beta} = \frac{[1.8\alpha\beta + 1.4332(7.332)\alpha + 1.3356(6.56)\beta + (7.332)(6.56)]K_{00}}{\alpha\beta + 7.332\alpha + 6.56\beta + (7.332)(6.56)} \quad (90)$$

while for the uniform load distribution case

$$K_{\alpha\beta} = \frac{[1.8\alpha\beta + 1.4(6.9053)(\alpha + \beta) + (6.9053)^2]K_{00}}{\alpha\beta + 6.9053(\alpha + \beta) + (6.9053)^2} \quad (91)$$

### Correlation of the Various Cases of Linearly Varying Axial Compression with the Uniform Axial Compression Case

The previous result which expresses the various curves of  $K_{\alpha\beta}$  in terms of the  $K_{00}$  curve, for several loading configurations, serves to emphasize and enforce the belief that there must be a general relationship between the various loading configurations. Since the prior section demonstrates that the simply supported case is the essential one (since  $K_{\alpha\beta}$  = a function of  $K_{00}$ ), attention will be focused upon the simply supported plate subjected to various linearly varying loading configurations. As has been mentioned previously, only in the simply supported case do the true aspect ratio and the effective aspect ratio coincide.

It can be seen from the trapezoidal, the triangular and the uniform load cases that the buckling coefficients, for the same boundary conditions, vary inversely with the loading profile fraction (L.P.F.). This loading profile fraction is the ratio of the area of the linearly varying loading distribution to the area of uniform distribution. Hence, the triangular distribution which is one-half of the uniform

distribution has a buckling coefficient  $K$  which is approximately twice the buckling coefficient for the uniform distribution. For the trapezoidal distribution which is  $3/4$  of a uniform distribution, the buckling coefficients are  $4/3$  times those of the uniform distribution. This relationship is quite satisfactory as long as the loading profiles do not contain any tension portions such as occur in pure or impure (unequal) bending. It has been noticed for many years, as mentioned by Gerard and Becker [156], that the buckling coefficients for pure bending with equal elastic lateral restraints were approximately six times those of uniform compression for the respective boundary conditions. No explanation has previously been advanced which can satisfactorily account for this situation. Obviously, the L.P.F. alone is not sufficient since it would predict buckling coefficients which were only four times those of uniform compression (in pure bending, the axial compression component, is  $1/4$  that of a uniform distribution; hence,  $1/L.P.L. = 4$ ). But as has been shown, the additional unknown factor must be identically one in the triangular and trapezoidal loading cases.

This predicament would still be unresolved if it were not for the fact that the various  $K_{\alpha\beta}$  can be expressed in terms of  $K_{00}$ . When the simply supported cases of triangular, trapezoidal, uniform and pure bending in-plane loadings are examined; the following fact becomes evident. The aspect ratio at which the minimum occurs in the first half wave of the buckling coefficient curve is the unknown correlation factor. This aspect ratio is one for the triangular, the trapezoidal, and the uniform cases and it is  $2/3$  for the case of pure bending. Therefore, the

uniform, triangular and trapezoidal cases are unchanged while for pure bending,  $1[(1/4) \times (2/3)] = 1/(1/6) = 6$  times the uniform buckling coefficient which is the proper value. This relationship was also applied to the other two cases of trapezoidal loading and one of impure bending given by Timoshenko [45] and was found to apply in those cases as well. Thus, the correlation between all known linearly varying axial compression cases and uniform compression may be expressed as follows:

The buckling coefficient of a linearly varying axial compression configuration is  $N$  times that of the corresponding uniform compression case where  $N$  is the reciprocal of the product of the loading profile fraction, L.P.F., and the aspect ratio at which the minimum occurs in the first half-wave of the buckling curve for the simply supported plate under linearly varying axial compression

$$N = \frac{1}{(\text{L.P.F.}) \times (\text{Aspect Ratio for the first half-wave minimum in the simply supported case})} \quad (92)$$

It may be noted that this aspect ratio at which the minimum occurs in the first half-wave is related to the buckle length for a long plate. A long, simply supported plate under a triangular, trapezoidal or uniform distribution of axial compression will buckle such that each half-wave has a length equal to the width of the plate (aspect ratio = 1). The long, simply supported plate subjected to pure bending has buckles whose length are  $2/3$  the width of the plate.

### Redefinition of the Loading Profile Parameter

The cases of linearly varying axial compression may now be put into a general form by using the rational function expressions for the buckling coefficients. In order to do this, it is necessary to redefine the value of  $\alpha_L$ . The loading profile parameter  $\alpha_L$  will now be defined as the ratio of the value of the lesser load to the value of greater load. Letting the lesser load value be " $a_L$ " and the greater load value be " $b_L$ ", the loading profile parameter now is  $\alpha_L^* = a_L/b_L$ . This new definition of  $\alpha_L$  has the property that its values now have a range including 0 and  $\infty$ . Furthermore, it can be used as a parameter against which may be plotted the values of the buckling coefficient,  $K$ , for the various cases. This plot is shown in Figure 47. It is seen from this plot that the various values of  $(K, \alpha_L^*)$  lie on a smooth curve.

A second curve of interest can be constructed with  $\alpha_L^*$  as one of its parameters. The other parameter is the coefficient,  $C$ , which is utilized in the rational function representations for  $K_{0,\beta}$  and  $K_{\alpha,0}$  for the various cases of linearly varying axial compression. It will be remembered that these coefficients appear in the final formula which relates a general  $K_{\alpha\beta}$  to  $K_{00}$ . Thus, by utilizing this curve, Figure 48, loading cases intermediate to the ones investigated here may also be approximated since the variation in the coefficient  $C$  may be obtained from the curve. This, therefore, constitutes a broader solution to linearly varying axial compression than any previous.

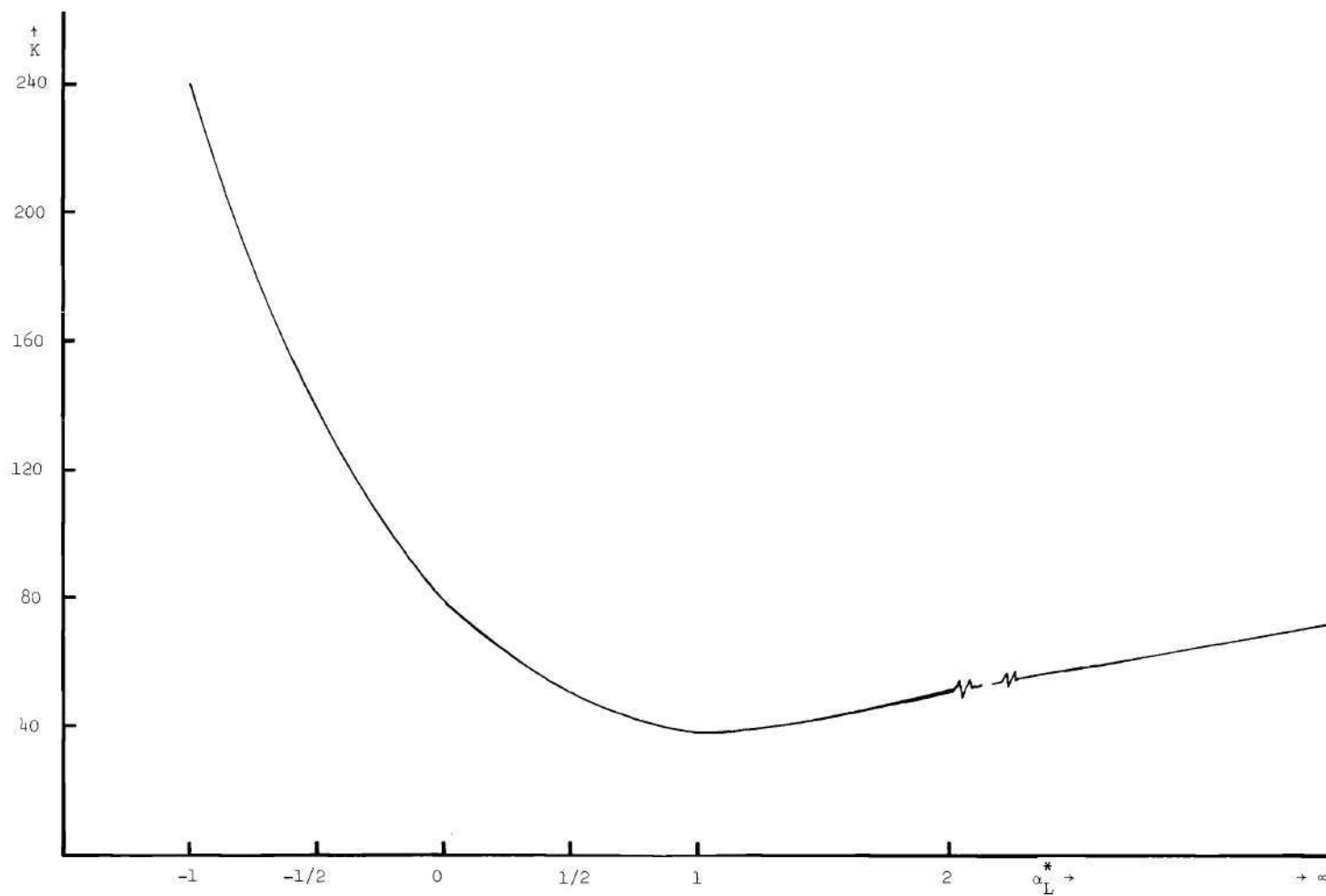


Figure 47.  $K$  versus  $\alpha_L^*$  Curve.



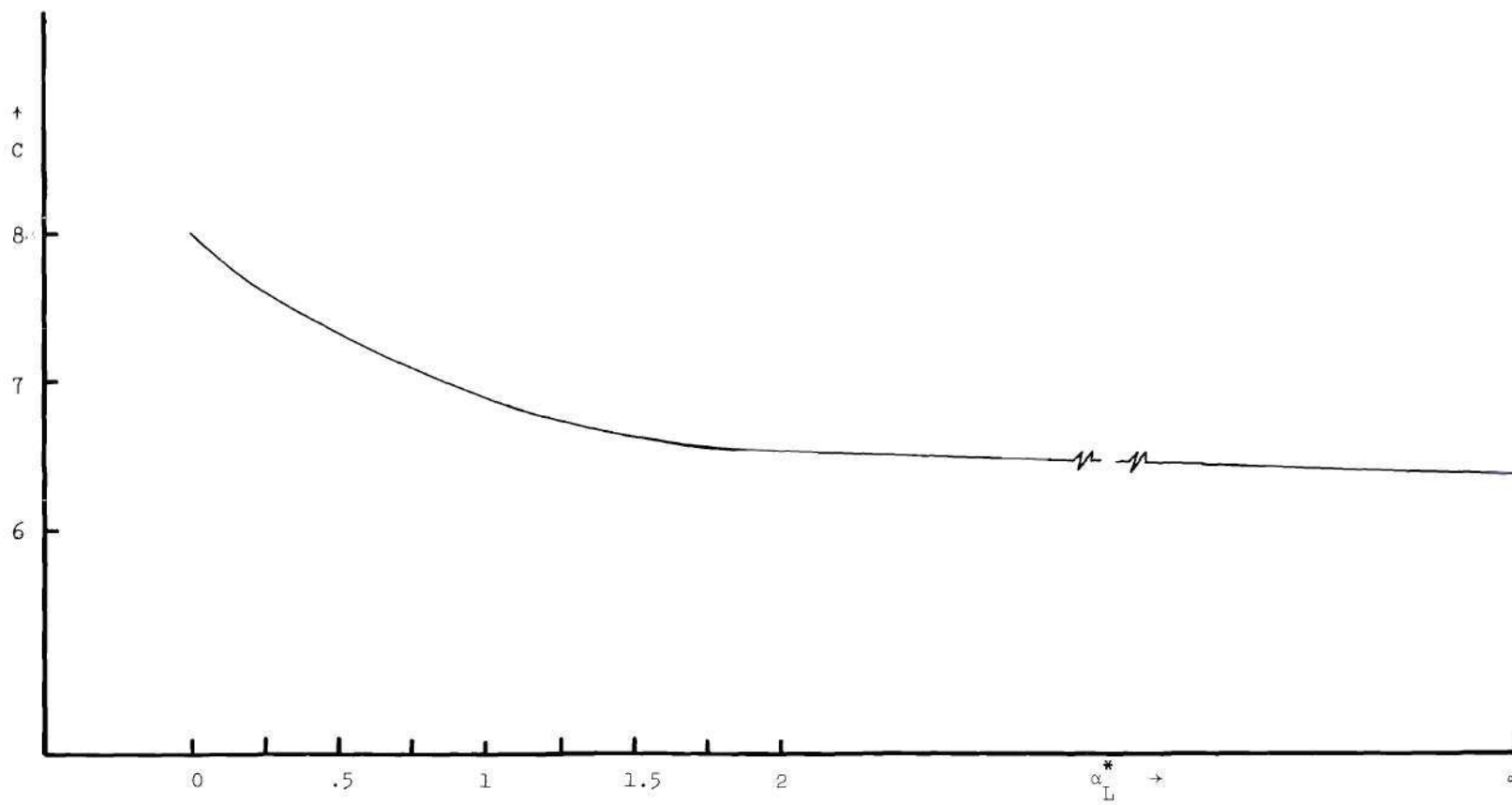


Figure 48.  $C$  versus  $\alpha_L^*$ .

## CHAPTER V

## CONCLUSIONS AND RECOMMENDATIONS

For almost a century experimentalists and theoreticians have studied the problem of plate instability. Today, due to refinement in test techniques and the general adoption of the Southwell method for interpreting data from stability tests, excellent agreement is attained between experimental results and theoretical predictions. Thus, we may have full confidence that the theory of plate stability rests on a secure foundation.

The use of the Raleigh Quotient has long been recognized as a sound way of obtaining an approximation to instability load levels. The problem in its use is, of course, the choice of the appropriate deflection function. The detailed study outlined in this thesis shows that without doubt the issues of plates under non-uniformly distributed axial compression can be satisfactorily treated provided the chordwise displacement is chosen to be of the same form as that for a beam, with analagous end restraints, subjected to a normal load of similar distribution to the compressive force. This method has been thoroughly verified from the solutions derived in the classic manner, and has been used to generate solutions to a number of problems not previously resolved; such as the plate with a triangular distribution of axial compression and equal or unequal elastic lateral restraints and the plate with a trapezoidal distribution of axial compression and equal or unequal elastic lateral restraints.

It has been found that both the old and the new data can be expressed by simple, general, accurate formula which give plate buckling coefficients in terms of boundary restraint character, loading conditions and the physical dimensions of the plate. Such simple formulae condense the normal plethora of charts and curves into a form more readily disseminated. These formulae, which are of rational function form, cover the variation in boundary restraint parameters from 0 to  $\infty$ , i.e., from simple support to fully fixed and avoid the need to interpolate which is necessary.

It is recommended that a concerted effort be made to use the general instability formulae with the similar expressions for vibration frequency which have been developed. This should lead to a practical method of assessing boundary restraint in realistic structures.

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## VITA

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Following graduation, Mr. Christian entered the U. S. Civil Service as an aerospace structural engineer at the Warner Robins Air Material Area, Robins A.F.B., Georgia. While in this capacity, he began work toward the Master of Science in Aerospace Engineering at the Georgia Institute of Technology in September 1968. He was graduated with the M.S.A.E. degree in June 1970.

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In September 1973, he began work with Combustion Engineering, Inc., in Chattanooga, Tennessee. He was employed in the Analytical Section of the Nuclear Components Department as a Senior Design Engineer on the Fast Flux Training Facility (FFTF) reactor vessel.

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